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ROUND-OFF STABILITY OF ITERATION PROCEDURES FOR SET-VALUED OPERATORS IN *b*-METRIC SPACES

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ABSTRACT

In this paper Ostrowski's type stability of iteration procedures for Banach set-valued operators in generalized Hausdorff metric spaces is considered.

Key Words and Phrases : Stability of iterations, set-valued Banach contraction operators.

Mathematics Subject Classifications (2001) : 47H10, 65D15, 41A25.

INTRODUCTION

Let us consider a metric space (X, d) and an operator $T: X \to X$. Problem of existence of fixed points of T leads to an iteration procedure $x_{n+1} = f(T, x_n)$. However, in computation, an approximate sequence $\{y_n\}$ is appplied istead of $\{x_n\}$ This leads to a concept of stability of iteration procedure with respect to T. The first stability result in metric spaces was proved by A.M. Ostrowski [5].

For any $x_0 \in X$, we put

$$x_{n+1} = f(T, x_n), \; n = 1, 2, \ldots . \tag{1}$$

Let $\{x_n\}$ be a sequence convergent to a fixed point $u \in X$ of T. Let $\{y_n\} \subset X$ be an arbitrary sequence, and set

$$\varepsilon_n = d(y_{n+1}, Ty_n), \; n = 0, 1, 2, \ldots . \tag{2}$$

The iteration procedure (1) is said to be T-stable (cf [4]) provided that

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$\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = u$.

Ostrowski's stability theorem is as follows.

Theorem 1 ([5], see also [3], [6]) Let (X, d) be a complete metric space and $T: X \to X$ a Banach contraction with constant g. Let $u \in X$ be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. Suppose that $\{y_n\}$ satisfying (2) is a sequence in X. Then, for $n = 0, 1, 2, \ldots$,

$$d(u, y_{n+1}) \le d(u, x_{n+1}) + g^{n+1} d(x_0, y_0) + \sum_{r=0}^{n} g^{n-r} \varepsilon_r. \tag{3}$$

Moreover,

$$\lim_{n\to\infty} y_n = u \quad iff \quad \lim_{n\to\infty} \varepsilon_n = 0 . \tag{4}$$

This result has been extended by Harder-Hicks [3], Rhoades [6], Singh -Chadha [8] and by authors of the paper [2] to single-valued and set-valued Banach operators.

In this paper we consider the problem of stability of iteration process for set-valued mappings in b-metric spaces.

SET-VALUED CONTRACTION MAPPINGS

Some mathematical problems, such as the problem of metrization of convergence with respect to measure, lead to a generalization of metric.

Now we recall (see S. Czerwik [1]) the idea of b-metric space.

Definition 1 ([1]). Let X be a set and $s \ge 1$ a given real number. A function $d: X \times X \to R_+$ is called a b-metric provided that for all $x, y, z \in X$,

$$d(x, y) = 0 \quad iff \quad x = y, \tag{5}$$

$$d(x, y) = d(y, x), \tag{6}$$

$$d(x, z) \le s[d(x, y) + d(y, z)]. \tag{7}$$

A pair (X, d) is called a b-metric space.

Note that, for $x \in X$ and $A \subset X$,

$$D(x, A) := \inf_{a \in A} d(x, a).$$

Definition 2 ([1]). For nonempty subsets A, B of a b-metric space (X, d), we define

$$H(A,B) := \begin{cases} \max\left\{ \sup\limits_{x \in A} D(x,B), \sup\limits_{y \in B} D(y,A) \right\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise} \end{cases}$$

By $CL(X)$ we denote the space of all nonempty closed subsets of X.

Theorem 2. ([1]). If (X, d) is a b-metric space, then the function $H : CL(X) \times CL(X) \to [0, \infty)$ is a generalized b-metric in $CL(X)$. Moreover, if (X, d) is a complete b-metric space, then $(CL(X), H)$ is also a complete b-metric space.

Now we can present the following fixed point theorem (see [1]).

Theorem 3. ([1]). Let (X, d) be a complete b-metric space. If $T: X \to CL(X)$ satisfies the inequality

$$H[Tx, Ty] \le \alpha d(x,y), \; x,y \in X, \text{ where} \tag{8}$$

$$0 \le \alpha < s^{-1}, \text{ then} \tag{9}$$

(i) for every $x_0 \in X$, there exist a sequence $\{x_n\} \subset X$ and $u \in X$ such that

$x_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$, and $\lim\limits_{n \to \infty} x_n = u$;

(ii) the point u is a fixed point of T, i.e. $u \in Tu$.

STABILITY RESULTS

S.L. Singh and V. Chadha [8] introduced the following definition of stability of iteration process for set-valued maps.

Let X be a metric space and $T: X \to CL(X)$. Let $x_0 \in X$ and

$$x_{n+1} \in f(T, x_n) \tag{10}$$

denote some iteration process. Let $\{x_n\}$ be convergent to a fixed point u of T and $\{y_n\}$ be an arbitrary sequence. Put

$$\varepsilon_n = H[y_{n+1}, f(T, y_n)],\ n = 0, 1, 2, \ldots .$$

Iteration process (10) is T-stable or stable with respect to T provided that

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{implies that} \quad \lim_{n\to\infty} y_n = u.$$

The following result has been proved in [2] (see also [7]).

Lemma. Let $\{\varepsilon_n\}$ be a sequence of nonegative real numbers. Let

$$s_n = \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r, \text{ where} \tag{11}$$

$0 \le \alpha < 1$. Then (12)

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \textit{iff} \quad \lim_{n\to\infty} s_n = 0. \tag{13}$$

Now we can prove the following result concerning the stability of iteration procedure for set-valued operators in b-metric spaces.

Theorem 4. Let (X, d) be a compete b-metric space and let $T: X \to CL(X)$ satisfy the inequlaity (8) with (9). Let $x_0 \in X$ and $\{x_n\}$ be an orbit for T at x_0, i.e. $x_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$, and $\{x_n\}$ converges to a fixed point u of T. Moreover, let $\{y_n\}$ be a sequence in X and set

$$\varepsilon_n = H(y_{n+1}, Ty_n),\ n = 0, 1, 2, \ldots .$$

Then

$$d(u, y_{n+1}) \le sd(u, x_{n+1}) + s(s\alpha)^{n+1} d(x_0, y_0) + s^2 \sum_{r=0}^{n} (s\alpha)^{n-r} \varepsilon_r \tag{14}$$

If, moreover, Tu is singleton, then

$$\lim_{n\to\infty} y_n = u \quad \text{iff} \quad \lim_{n\to\infty} \varepsilon_n = 0. \tag{15}$$

Proof. In view of the fact that $x_{n+1} \in Tx_n$ for $n = 0, 1, \ldots$ and the definition of H, we get $d(x_{n+1}, y_{n+1}) \leq H(Tx_n, y_{n+1})$.

Now, since H is a b-metric, so for any $A, B, C \in CL(X)$ we have

$$H(A, C) \leq s[H(A, B) + H(B, C)].$$

Therefore by (8) we obtain, for any nonegative integer n,

$$d(x_{n+1}, y_{n+1}) \leq H(Tx_n, y_{n+1}) \leq s[H(Tx_n, Ty_n) + H(Ty_n, y_{n+1})] \leq s\alpha\, d(x_n, y_n) + s\varepsilon_n$$

$$\leq s\alpha[s\alpha d(x_{n-1}, y_{n-1}) + s\varepsilon_{n-1}] + s\varepsilon_n \leq (s\alpha)^2 d(x_{n-1}, y_{n-1}) + s[s\alpha\varepsilon_{n-1} + \varepsilon_n].$$

Appyling the induction principle, we come to the following inequality

$$d(x_{n+1}, y_{n+1}) \leq (s\alpha)^{n+1} d(x_0, y_0) + s\sum_{r=0}^{n} (s\alpha)^{n-r} \varepsilon_r$$

Consequently,

$$d(u, y_{n+1}) \leq sd(u, x_{n+1}) + sd(x_{n+1}, y_{n+1})$$

$$\leq sd(u, x_{n+1}) + s(s\alpha)^{n+1} d(x_0, y_0) + s^2 \sum_{r=0}^{n} (s\alpha)^{n-r} \varepsilon_r,$$

which is just the inequality (14).

To establish (15), assume that $\lim_{n\to\infty} y_n = u$ Then we get the sequence of inequalities :

$$\varepsilon_n = H(y_{n+1}, Ty_n) \leq sH(y_{n+1}, u) + sH(u, Ty_n) \leq sd(y_{n+1}, u) + s[sH(u, Tu) + sH(Tu, Ty_n)]$$

$$\leq sd(y_{n+1}, u) + s^2\alpha d(u, y_n).$$

But $d(u, y_n)$ and $d(u, y_{n+1})$ tend to zero as $n \to \infty$, therefore $\varepsilon_n \to 0$ as $n \to \infty$.

To prove the converse implication, suppose that $\varepsilon_n \to 0$ as $n \to \infty$. Taking into account that $x_n \to u$ and $0 \le s\alpha < 1$, we get $d(u, x_{n+1}) \to 0$ and , $s(s\alpha)^{n+1} d(x_0, y_0) \to 0$ as $n \to \infty$. Appyling Lemma to the last term of the inequality (14), we see that

$$\lim_{n\to\infty} \sum_{r=0}^{n} (s\alpha)^{n-r} \varepsilon_r = 0,$$

yielding $\lim_{n\to\infty} y_n = u$. This completes the proof.

Remark 1. Inequality (14) gives some estimation of $d(u, y_n)$.

2. It is easy to present examples which can show that T may have more than one fixed point.

3. Relation (15) implies Ostrowski's type stability of Picard sequence of succestive approximations for set-valued contractions in b-metric spaces.

Applying an idea of [8], we peresent another throrem on stability for set-valued mappings.

Theorem 5. Let the hypotheses of Theorem 4 hold, where ε_n and the inequality (9) are replaced by the following.

$\varepsilon_n = d(y_{n+1}, u_n),\ u_n \in Ty_n,\ n = 0,1,2, \ldots.$

and

$$0 \le \alpha < s^{-2}. \tag{16}$$

Then

$$d(u, y_{n+1}) \le sd(u, x_{n+1}) + sg^{n+1} d(x_0, y_0) + s^2 \sum_{r=0}^{n} g^{n-r} (sH_r + \varepsilon_r), \tag{17}$$

where $g = s^2\alpha$, $H_r = H(x_{r+1}, Tx_r)$. Moreover, if Tu is singleton, then

$$\lim_{n\to\infty} y_n = u \text{ iff } \lim_{n\to\infty} \varepsilon_n = 0. \tag{18}$$

Proof. First we show that

$$d(x_{n+1}, y_{n+1}) \le g^{n+1} d(x_0, y_0) + s\sum_{r=0}^{n} g^{n-r}(sH_r + \varepsilon_r) \text{ for } n = 0, 1, 2, \ldots . \tag{19}$$

For $n = 0$, we have (in view of our assumptions),

$$d(x_1, y_1) \le s[d(x_1, u_0) + d(u_0, y_1)] \le sH(x_1, Ty_0) + s\varepsilon_0$$

$$\le s[sH(x_1, Tx_0) + sH(Tx_0, Ty_0)] + s\varepsilon_0 \le s^2 H_0 + s^2 \alpha\, d(x_0, y_0) + s\varepsilon_0,$$

i.e. the inequality (19) (for n=0). Now, applying (19) we get for n+2 :

$$d(x_{n+2}, y_{n+2}) \le s[d(x_{n+2}, u_{n+1}) + d(u_{n+1}, y_{n+2})] \le sH(x_{n+2}, Ty_{n+1}) + s\varepsilon_{n+1}$$

$$\le s[sH(x_{n+2}, Tx_{n+1}) + sH(Tx_{n+1}, Ty_{n+1})] + s\varepsilon_{n+1} \le gd(x_{n+1}, y_{n+1}) + s(sH_{n+1} + \varepsilon_{n+1})$$

$$\le g^{n+2} d(x_0, y_0) + s\sum_{r=0}^{n} g^{n+1-r}(sH_r + \varepsilon_r) + s(sH_{n+1} + \varepsilon_{n+1})$$

$$\le g^{n+2} d(x_0, y_0) + s\sum_{r=0}^{n+1} g^{n+1-r}(sH_r + \varepsilon_r)$$

This, in view of induction principle, proves the imequality (19) for all $n \ge 0$. Consequently, the relation (17) follows directly from

$$d(u, y_{n+1}) \le sd(u, x_{n+1}) + sd(x_{n+1}, y_{n+1}).$$

To prove (18), assume $\lim_{n\to\infty} y_n = u$. Then $\varepsilon_n = d(y_{n+1}, u_n) \le H(y_{n+1}, Ty_n)$ and we can repeat the arguments used to prove Theorem 4.

Now conversly, let $\lim_{n\to\infty} \varepsilon_n = 0$. Since $Tu = u$, we get

$$H_r = H(x_{r+1}, Tx_r) \le sH(x_{r+1}, Tu) + sH(Tu, Tx_r) \le sd(x_{r+1}, Tu) + s\alpha d(u, x_r)$$

$$\le sd(x_{r+1}, u) + s\alpha d(u, x_r).$$

Therefore, $H_r \to 0$ as $r \to \infty$. Now, because $0 \le g < 1$ and $sH_r + \varepsilon_r \to 0$ as $r \to \infty$, in view of the inequality (17) and Lemma, we come to the conclusion that $\lim_{n\to\infty} y_n = u$. This completes the proof.

Remark 4. If $s = 1$ and T is single-valued operator, then from Theorems 4 and 5, we get Ostrowski's Theorem 1.

5. If $s = 1$, from Theorems 4 and 5, we can derive results of Singh and Chadha [8].

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ASTRONOMICAL CONSTANTS

Ramesh Chand* and Prabhakar Pradhan**

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ABSTRACT

The intent of this paper is to explain the Astronomical Constants given by Bhāskarācārya (1150 A.D.). The planetary position can be computed for any day with the help of Astronomical Constants.

INTRODUCTION

In his *Grahagaṇita* (C. 1150 A.D.) *Bhāskarācārya* II (b. 1114 A.D.) has given several computational methods for the determination of planetary position at a particular place and time (See [2], [5] and [15]). The classical presentation of *Bhāskarācārya* II, highly respected formulae are aparently not very friendly to a modern astronomer. *Bhāskarācārya* II has used certain constants which we shall call planetary constants or astronomical constants in computing longitudes of planets. In this paper we attempt a systematic presentation of these astronomical constants which can be used very conveniently be a sophomore-astronomer for computing planetary position.

ASTRONOMICAL CONSTANTS

In the following stanza of *Grahagaṇita Bhāskarācārya*-II gives the astronomical constants (See [2], [5] and [15]) :

दिग्भिः 101461 नगाष्टनगभूतिथिभिः क्रमेण 151787 ।
देवाष्टखाङ्कशशिभिः 190833 च रसाग्निवेद-
सिद्धैः 24436 खखाब्धिदहनाभ्रयमेन्दुभिश्च 1203400 ।।
भूपाब्धिलोचनरसैः 62416 खखखाभ्रनन्द
नन्दाश्विभिः 2990000 गगनखाभ्रगजाङ्कनागैः 898000 ।
खाभ्राष्ट पङ्क्तिजघृतिप्रमितैः 1886800 च भक्ताद्
भागादिकानि हि फलानि खेः राकाशात्।।
विधोः फलं खाश्विगुणं विधेयं ग्रहध्रुवाः स्वस्वफलैः समेताः।
ते वा भवन्ति द्युचराः क्रमेण भागादिकः स्यात् फलमेव भानुः।।

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According to Somayaji [2,p. 68] :

The *ahargaṇa* multiplied by 100000 and divided successively by 101461, 151787, 190833, 24436, 1203400, 62416, 2990000, 898000, 1886800 gives the respective planetary positions of the sun, the moon, Mars, *Budha* (Mercury), Jupiter Venus, Saturn, apogee of the moon and lunar node. In the case of the moon, however the result is to be multiplied by 20. The results in degrees added to the *Dhruvakās* give their positions.

Arakasomyaji [2] illustrated that the degrees covered by the planets etc. in A days are $\frac{Rx360xA}{M}$ where R is the number of sidereal revolutions of a planet in a *Kalpa,* M the number of mean solar days in a *Kalpa,* and A the *ahargaṇa.*

$$Thus\ \frac{Rx360xA}{M} = \frac{Rx360xAx100000}{Mx100000} = \frac{Ax100000}{\frac{Mx100000}{Rx360}}$$

Then $\frac{Mx100000}{R}$ is found for every planet. In the case of the moon, however a multiplier 20 also is used, because the moon has quicker motion.

Constant for the sun

$$\frac{Mx100000}{Rx360} = \frac{1577916450000x100000}{4320000000x360} = 101460$$

Constant for the moon

$$= \frac{1577916450000x20}{57753300000x360} = 151787$$

Example 1 : Compute the Planetary positions on May 25, 1993.

The mean longitude of the sun on May 25, 1993

$$= \frac{Ax100000}{101461} \text{Degrees}$$

$$= \frac{1860667x100000}{190833} Degrees + Dhravak\bar{a}$$

$$= \frac{1860667x100000}{360x190833} \text{Re}\, volution + Dhruvak\bar{a}$$

$= 5094.0947$ Revolution/ 1 Rāsi /4°/5'/ 57.768"

The mean longitude of Moon on May 25, 1993

$$= \frac{1860667 \times 100000 \times 20}{151787} \text{ Degrees}$$

$$= \frac{1860667x100000x20}{360x151787} \text{Re volutions}$$

$= 68102.268$ Re volutions $= 68102$ Revolutions/3 Rās'i / 6° / 19'/41.088"

The mean longitude of Mars on May 25, 1993

$$= \frac{1860667x100000}{190833} \text{Degrees} + \text{Dhruvak}\bar{\text{a}}$$

$$= \frac{1860667x100000}{360x190833} \text{Re volution} + \text{Dhruvak}\bar{\text{a}}$$

$= 2708.3992$ Revolutions + 11 Rasi /29° / 3'/ 5"

$= 2709$ Revolutions/ 4 Rās'i / 22° / 46'/ 33.2"

The mean longitude of Mercury on May 25, 1993

$$= \frac{1860667 \times 100000}{360 \times 24436} \text{ Revolution} + \text{Dhruvak}\bar{\text{a}}$$

$= 21151.25$ Revolutions+11 Rās'i / 29°/ 27'/ 36"

$= 21151$ Revolutions/ 89° / 59' / 30.192" + 11 Rās'i / 29° / 27' / 36"

$= 21151$ Revolution /3 Rās'i / 29° / 27' / 6.192"

Constant for Mars

$$= \frac{1577916450000x100000}{2296828522x360} = 190833$$

Constant for Mercury

$$= \frac{1577916450000 x 100000}{17936998984 x 360} = 244436$$

Constant for Jupiter

$$= \frac{1577916450000 x 100000}{364226455 x 360} = 1203400$$

Constant for Venus

$$= \frac{1577916450000 x 100000}{7022389492 x 360} = 62416$$

Constant for Saturn

$$= \frac{1577916450000 x 100000}{1465672980 x 360} = 2990504$$

Note that *Bhāskarācārya* obtained the constant 2990000 for Satrun in place of 2990504.

Constant for the apogee of the moon

$$= \frac{1577916450000 x 100000}{488105858 x 360} = 897981$$

Note that Bhāskarācārya obtained the constant 898000 for the apogee of the moon in place 897981.

Constant for the lunar node

$$= \frac{1577916450000 x 100000}{232311168 x 360} = 1886737$$

Note that Bhāskarācārya obtained the constant 1886800 for the lunar node in place of 1886737

The mean longitude of Jupiter on May 25, 1993

$$= \frac{1860667 \times 100000}{1203400 \times 360} \text{ Revolution} + \text{Dhruvak}\bar{a}$$

= 429.49306 Revolution + 11 Rās'i / 29° / 27' / 36"

= 430 Revolutions / 5 Rās'i / 26° / 57' / 41. 76"

The mean longitude of Venus on May 25, 1993

$$= \frac{1860667 \times 100000}{62416 \times 360} \text{Re}\,volutions + Dhruvak\bar{a}$$

= 8280.7605 Revolutions + 11 Rās'i / 28° / 42' / 14"

= 8281 Revolutions / 9 Rās'i / 2° / 28' / 0.2176"

The mean longitude of Saturn on May 25, 1993

$$= \frac{1860667 \times 100000}{299050 \times 360} \text{Re}\,volutions + Dhruvak\bar{a}s$$

= 172.83105 Revolutions + 11 Rās'i / 28° / 46'/ 34"

= 172 Revolutions / 299° / 10'/ 40.8"+ 11 Rās'i / 28° / 46' / 34"

= 173 Revolutions / 3 Rās'i / 27° / 57' / 14.8"

The mean longitude of the moon's apogee on May 25, 1993

$$= \frac{1860667\text{x}100000}{897981\text{x}360} \text{Re volutions} + \text{Dhruvak}\bar{a}$$

= 575.57114 Revolutions + 4 Rās'i / 5°/ 29' / 46"

= 575 Revolutions / 205° / 36' / 37.44" + 4 Rās'i / 5° / 29' / 46"

= 575 Revolutions / 11 Rās'i / 1° / 6' / 23.44"

The mean longitude of the lunar node on May 25, 1993

$$= \frac{1860667\text{x}100000}{1886737\text{x}360} \text{Re volutions} + \text{Dhruvakas}$$

= 273.93958 Revolutions + Dhruvakā

= 273.0 Revolutions / 338^0 / 14' / 57.79248" + 5 Rās'i / 3^0 / 12' / 58"

= 274 Revolutions / 4 Rās'i / 11^0 / 27' / 55.79248"

CONCLUSION

It appears that the methods suggested by Bhāskarācārya - II in his Grahaganita for the computation of planetary positions are identical to each other.

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SEMI C^v- REDUCIBLE FINSLER SPACES

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ABSTRACT

Our purpose is to generalise the concept of C^v- reducibility under the certain restriction of semi C-reducible Finsler Spaces.

1991 Mathematical subject classification : *30B30, 40C05, 40G10, 4A99, 53 C 60.*

Key words and Phrases : Finsler space, semi C^v-reducible Finsler space.

INTRODUCTION

Let F^n be an n dimensional Finsler space with fundamental function $L(x,y)$, the fundamental metric tensor g_{ij} and the angular metric tensor h_{ij}.

It has been studied in [6], a special Finsler space, called C^v-reducible Finsler space characterized by

$$LC_{ijk}\Big|_l = -\frac{1}{n+1} \qquad [h_{ijk}C_l + h_{jkl}C_i + h_{kli}C_j + h_{lij}C_k] \tag{1.1}$$

where $\quad h_{ijk} = h_{ij}l_k + h_{jk}l_i + h_{ki}l_j,\ \ C_i = g^{jk}C_{ijk}$

In the same paper we have also been seen that if the Cartan v-derivative of $h(hv)$ torsion tensor can be written in the form

$$LC_{ijk}\Big|_l = A_{ijk}B_l + A_{jkl}B_i + A_{kli}B_j + A_{lij}B_k, \tag{1.2}$$

where A_{ijk} is symmetric in its indices but not proportional to C_{ijk}; then we have $A_{ijo} \neq 0,\ A_{ioo} = 0$ and $B_o = 0$ where 'o' indicates contraction by y^i. If we take $B_i = C_i$ (contracted torsion tensor) then we get.

Definition (1) :

A non-Riemannian Finsler space F^n; for which the Cartan v-derivative of $h(hv)$ torsion tensor can be written as

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$$LC_{ijk}|_l = A_{ijk}C_l + A_{jkl}C_i + A_{kli}C_j + A_{lij}C_k$$

where A_{ijk} are not proportional to C_{ijk} and satisfying $A_{ijo} \neq o$, $A_{ioo} = o$; is called a quasi C^v-reducible Finsler space.

If we take special form of A_{ijk} as

$$A_{ijk} = \lambda\, h_{ijk} + \mu\, \pi_{(i,j,k,)}\, (C_iC_jl_k)$$

where $\pi_{(i,j,k)}$ denotes cyclic permutation of i,j,k and summation. Then *(1.3)* gives

$$LC_{ijk}|_l = \lambda\, \pi_{(i,j,k,l)}\, [h_{ijk}C_l] + 3\,\mu\, \pi_{(i,j,k,l)}[C_iC_jC_kl_l] \tag{1.4}$$

where $\pi_{(i,j,k,l)}$ denotes cyclic permutation of indices i,j,k,l and summation.

Contracting *(1.4)* by y^l and again by g^{jk} we get

$$(n+1)\,\lambda + 3C^2\,\mu + 1 = 0 \tag{1.5}$$

if we choose $\lambda = -p/(n+1)$, $3\mu = -\frac{q}{C^2}$ then (1.5) leads to $p+q=1$. For these particular values of λ and μ we define a special Finsler space as follows.

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Definition (2) :

A non-Riemannian Finsler space F^n is called semi C^v-Reducible Finsler space, if the Cartan *v*-derivative of *h (hv)* torsion tensor C_{ijk} can be written as

$$LC_{ijk|l} = -\frac{p}{n+1}\,[h_{ijk}C_l + h_{jkl}C_i + h_{kli}\, l_j/h_{lij}\, C_k]$$

$$-\frac{q}{C^2}\,[C_iC_jC_kl_l + C_jC_kC_ll_i + C_kC_lC_il_j + C_lC_iC_jl_k] \tag{2.1}$$

where $p+q=1$

Remark (1): In case of $q = o$ the space reduces to C^v-reducible Finsler space which has been studied in [6].

proposition (1): for n ≥ 3, every semi C^v-reducible Finsler space is semi C-reducible. The necessary and sufficient condition for a semi C-reducible Finsler space with constant coefficients to be semi C^v-reducible is

$$LC_i|_j + C_i l_j + C_j l_i = 0 \; provided \; p \neq 5/2$$

Proof : Contracting equation (2.1) by y^l we get

$$C_{ijk} = \frac{p}{n+1}\,[h_{ij}C_k + h_{jk}C_i + h_{ki}\,C_j] + (q/C^2)\,C_iC_jC_k \tag{2.2}$$

A Finsler space F^n $(n \geq 3)$ satisfying the condition (2.2) with $p+q=1$ is semi C-reducible [4].

For second part of the proposition, differentiate *(2.2)* covariantly with respect to y^j, and using the following relations; which has already been shown in *[5]* for a semi C-reducible Finsler space with constant coefficient ;

$$C_i|_j = \alpha h_{ij} - \frac{(C_i l_j + C_j l_i)}{L} + \beta C_i C_j, \text{ where } \beta = \frac{(n+1)qa}{pC^2}$$

and
$$C^2|_i = 2(n+1-np)\alpha\frac{(C_i)}{p} - \frac{(2C^2 l_i)}{L}, h_{ij}|_l = \frac{h_{ij}l_l - h_{ijl}}{L} \tag{2.3}$$

where $\alpha = \dfrac{-pC^i|_i}{(n+1-2p)}$ or $\dfrac{-C^2|_i C^i}{2(n-2)C^2}$ according as $p \neq \dfrac{n+1}{2}$ or $p = \dfrac{n+1}{2}$;

we get

$$C_{ijk}\Big|_l = -\frac{p}{L(n+1)}\left[h_{ijk}C_l + h_{jkl}C_i + h_{kli}C_j + h_{lij}C_k\right]$$

$$-\frac{q}{LC^2}\left[C_iC_jC_kl_l + C_jC_kC_ll_i + C_kC_lC_il_j + C_lC_iC_jl_k\right]$$

$$+\frac{\alpha p}{n+1}[h_{ij}h_{kl} + h_{jk}h_{il} + h_{ki}h_{jl}]$$

$$+\frac{q\alpha}{C^2}[h_{ij}C_kC_l + h_{jk}C_iC_l + h_{ki}C_jC_l + h_{kl}C_iC_j + h_{jl}C_iC_k + h_{il}C_jC_K]$$

$$+q\left[\frac{3(n+1)q\alpha}{pC^4}-\frac{2(n+1-np)\alpha)}{pC^4}\right]C_iC_jC_kC_l \tag{2.4}$$

If the space is semi C^v-reducible the last three terms on R.H.S. of above equation must vanish. So,

$$\frac{\alpha p}{n+1}[h_{ij}h_{kl}+h_{jk}h_{il}+h_{ki}h_{jl}]+\frac{q\alpha}{C^2}[h_{ij}C_kC_l+h_{jk}C_iC_l+h_{ki}C_jC_l+h_{kl}C_iC_j$$

$$+h_{jl}C_iC_K+h_{il}C_jC_k]+\frac{q\alpha}{pC^4}[3(n+1)q-2(n+1-np)]\,C_iC_jC_kC_l=0 \tag{2.5}$$

Contracting above equation by g^{ij} and g^{kl} respectively we get

$$\alpha(-2p+n+1)=0 \tag{2.6}$$

which imply either $\alpha = 0$ or $p=\frac{n+1}{2}$

If $\alpha \neq 0$ we have $p=\frac{n+1}{2}$ For this value of p contracting (2.5) by $C^iC^jC^kC^l$ we have

$$\alpha\, C^4\,(n\text{-}4)\,(n\text{-}2)=0 \tag{2.7}$$

For a non-Riemannian semi C-reducible Finsler space $(n \geq 3)$ F^n satisfying (2.7) we have $n = 4$. Thus if we assume $n \neq 4$ i.e. $p \neq 5/2$ we have $\alpha = 0$ from (2.6) and (2.7). It is obvious from (2.4) that $a = 0$ is also sufficient for the given space to be semi C^v-reducible.

From (2.3) it can be easily seen that

$$\alpha=0 \Leftrightarrow LC_i\,|_j = -(C_il_j+C_jl_i)$$

Remark (2): Contracting equation (2.1) by g^{kl} we have,

In a semi C^v-reducible Finsler space the equation $LC_i\,|_j+C_il_j+C_jl_i=0$ holds good.

proposition (2) : The indicatrized tensors T_{ijkl} of $C_{ijk}\,|_l$ and T_{ijklm} of $S_{ijkl}\,|_m$ vanishes identically in a semi C^v-reducible Finsler space.

Proof : The indicatrized Tensor T_{ijkl} of $C_{ijk}|_l$ can be written as [2]

$$T_{ijkl} = L\, C_{ijk}|_l + C_{ijk} l_l + C_{jkl} l_i + C_{kli} l_j + C_{lij} l_k$$

For a semi C^v-reducible Finsler space on substituting the values of $C_{ijk}|_l$ and C_{ijk} etc, we have $T_{ijkl} = 0$

Further the indicatrized tensor T_{ijklm} of $S_{ijkl}|_m$ can be written as [7]

$$T_{ijklm} = T_{ilhm} C^h_{jk} + T_{jkhm} C^h_{il} - T_{ikhm} C^h_{jl} - T_{jlhm} C^h_{ik}$$

Which shows vanishing of the T-tensor imply that the tensor T_{ijklm} also vanishes.

Proposition (3) : In a non-Riemannian semi C^v-reducible Finsler space F^n, $(n>2)$ the tensor $C_{ijk}|_l|_m$ is not symmetric.

Proof : For a semi C^v-reducible Finsler space we have

$$L\, C_{ijk}|_l = -\frac{p}{n+1}\,[h_{ijk} C_l + h_{jkl} C_i + h_{kli} C_j + h_{lij} C_k]$$

$$-\frac{q}{C^2}\,\pi_{(i,j,k,l)}\,[C_i C_j C_k l_l] \tag{2.8}$$

Differentiating (2.8) covariantly with respect to y^m and using the fact $LC_i|_j = -(C_i l_j + C_j l_i)$ and $LC^2|_i = -2C^2 l_i$ in a semi C^v-reducible Finsler space, we have

$$L^2\, C_{ijk}|_l|_m = \frac{2p}{n+2}\,[(h_{ijkl} C_m + h_{jklm} C_i + h_{klmi} C_j + h_{lmij} C_k + h_{mijk} C_l]$$

$$+ (h_{ij} h_{kl} + h_{jk} h_{li} + h_{ki} h_{jl}) C_m]$$

$$+ \frac{2q}{C^2}\, l_m (C_i C_j C_k l_l + C_j C_k C_l l_i + C_k C_l C_i l_j + C_l C_i C_j l_k)$$

$$+ C_m (C_j C_k l_i l_l + C_i C_k l_l l_j + C_i C_j l_l l_k + C_k C_l l_i l_j + C_j C_l l_k l_i + C_i C_l l_j l_k)]$$

where $h_{ijkl} = h_{ijk} l_l - L\, h_{ijk}|_l$, which is symmetric in its all indices.

If the tensor $C_{ijk}|_l|_m$ is symmetric we have

$$L^2 (C_{ijk}|_l|_m - C_{ijk}|_m|_l) = 0, \text{ implies}$$

$$\frac{2p}{n+1}\,[(h_{ij}h_{kl}+h_{jk}h_{li}+h_{ki}h_{jl})\,C_m-(h_{ij}h_{km}+h_{jk}h_{mi}+h_{ki}h_{jm})C_l]$$

$$+\frac{2q}{C^2}[l_m\pi_{(i,j,k,l)}(C_iC_jC_kl_l)]-l_l\pi_{(i,j,k,m)}(C_iC_jC_kl_m)$$

$$+C_m(C_jC_kl_il_l+C_iC_kl_lj_j+C_iC_jl_ll_k+C_kC_ll_il_j+C_jC_ll_kl_i+C_iC_ll_jl_k)$$

$$-C_l(C_jC_kl_il_m+C_iC_kl_ml_j+C_iC_jl_ml_k+C_kC_ml_il_j+C_jC_ml_kl_i+C_iC_ml_jl_k)]=0$$

Contracting above equation by g^{ij} and further by g^{kl} we have

$$p\,[n\text{-}2]\,C_m=0$$

Since $p\neq 0$, $n>2$ we have $C_m=0$

and so the space becomes Riemannian. Hence for $n>2$, In a non-Riemannian semi C^v-reducible Finsler space $C_{ijk}|_l|_m$ is not symmetric.

Theorem (1)

A semi C^v-reducible Finsler space is Berwald iff $C_i|_o$ vanishes identically.

Proof :

For a semi C^v-reducible Finsler space.

$$LC_{ijk}|_l=-\frac{p}{(n+1)}\,[h_{ijk}C_l+h_{jkl}C_i+h_{kli}\,C_j+h_{lij}\,C_k]$$

$$-\frac{p}{(n+1)}\pi_{(i,j,k,l)}\,[C_iC_jC_kl_l] \tag{2.9}$$

Differentiating (2.9) covariantly with respect to x^m and contracting by y^m we get

$$LC_{ijk}|_{l|o}=\pi_{(i,j,k,l)}\,[C_i|_o(-\frac{p}{n+1}h_{jkl}-\frac{q}{C^2}(C_jC_kl_l+C_kC_ll_j+C_jC_ll_k)]$$

$$+\frac{2q}{C^4}\,C^rC_r|_oC_iC_jC_kl_l] \tag{2.10}$$

Further if $C_{ijk}\big|_{l|_o} = 0$ contracting (2.10) by g^{ij} we have

$$C_{k|_o}l_l + C_{l|_o}l_k = o \Leftrightarrow C_{l|_o} = 0$$

Also from (2.10) $C_{l|_o} = 0 \Rightarrow C_{ijk|l|_o} = 0$

Since Berwald space is characterised by $C_{ijk|l} = 0$ or $C_{ijk}\big|_{l|_o} = 0$ [1],

Hence in a semi C^v-reducible Finsler space Berwald spaces are characterised by $C_{i|_o} = 0$

Theorem (2)

The necessary and sufficient condition, for a two dimensional Finsler space to be semi C^v-reducible, is the main scalar be function of position alone.

Proof

It is well known that the h (hv) torsion tensor of a two dimensional Finsler space can be written as

$$LC_{ijk} = I\, m_i m_j m_k \tag{2.11}$$

where m_i is the unit vector along contracted torsion tensor C_i. Differentiating (2.11) covariantly with respect to y^l we have

$$LC_{ijk}\big|_l = -\frac{p}{3}[h_{ijk}C_l + h_{jkl}C_i + h_{kli}C_j + h_{lij}\, C_k]$$

$$-\frac{q}{C^2}[\pi_{(i,j,k,l)}\, C_iC_jC_kl_l] + \frac{\partial I}{\partial y l}\, m_i m_j m_k$$

here $p + q = 1$

The above equation imply that the necessary and sufficient condition for a two dimensional Finsler space to be semi C^v-reducible is, I be the function of position only.

Theorem (3)

For $n \geq 5$, a semi C^v-reducible Finsler space is equivalent to a S-4 like Finsler space with T-condition.

Proof :

For a semi C^v-reducible Finsler space we have from propositon (1),

$$C_{ijk} = \frac{p}{n+1}[h_{ij}C_k + h_{jk}C_i + h_{ki}C_j] + \frac{q}{C^2} C_iC_jC_k \tag{2.12}$$

So, the v-curvature tensor for a semi C^v-reducible Finsler space; which has already been shown in [4], and considering every semi C^v-reducible Finsler space is semi C-reducible; can be written as

$$L^2S_{ijkl} = h_{ik} M_{jl} + h_{jl} M_{ik} - h_{il}M_{jk} - h_{jk}M_{il}$$

Where the symmetric tensor M_{ij} is given by

$$\frac{M_{ij}}{L^2} = -\frac{(pC)^2}{2(n+1)^2}h_{ij} - \left[\frac{p^2}{(n+1)^2} + \frac{pq}{n+1}\right]C_iC_j$$

So, the space is S-4 like, T- condition is obvious from proposition (2).

From Theorem [4], Since any S-4 like Finsler space is semi C-reducible provided T-tensor vanishes. Further any semi C-reducible Finsler space with T-condition is semi C^v-reducible provided $p \neq 5/2$. The converse of theorem is proved.

Theorem (4)

The necessary and sufficient condition for a three dimensional Finsler space to be semi C^v-reducible is the main scalars satisfy the following conditions.

$J = 0$, H and I are function of position only such that $H = I + I^{-1}$, together with v-connection vector vanishes identically.

Proof:

It is well known that for a three dimensional Finsler space, the h (hv) torsion tensor can be wartime as

$$LC_{ijk} = H\, m_im_jm_k + J\pi_{(i,j,k)}\left(m_in_jn_k\right) + I\pi_{(i,j,k)}\left[m_im_jn_k\right] + Jn_in_jm_k \tag{2.13}$$

It is evident that a semi C^v- reducible Finsler space is quasi C-reducible. It has been seen in [5] that a three dimensional Finsler space satisfy T condition iff v-connection

vector v_i vanishes and all the main scalars H, I, J are function of position only. Further these scalars satisfy the relation $2J^2+I^2-HI=-1$. Also for a quasi C-reducible Finsler space $J=0$, so we get

$$H = I + I^{-1}$$

To prove sufficient part differentiate (2.8) covariantly with respect to y^1 and using the given condition we have

$$LC_{ijk|l} = -\pi_{(i,j,k,l)}\,[l_i(\frac{H}{L} m_j m_k m_l + \frac{I}{L} m_j n_k n_l + \frac{I}{L} m_k n_l n_j + \frac{I}{L} m_l n_j n_k)]$$

$$=-\frac{I}{CL}[\pi_{(i,j,k,l)} h_{ijk} C_l] - \frac{(H-3I)}{LC^3}\pi_{(i,j,k,l)}(C_i C_j C_k l_l)$$

If we put $p=\frac{4I}{LC}, q=\frac{H-3I}{LC}$ such that $p+q=1$, we have the form of a semi C^v-reducible Finsler space.

Remark (3) : Since for a three dimensional Finsler space $LC=H+I$ we have if a three dimensional Finsler space is semi C^v-reducible

$$2I^2 - (LC)\,I + 1 = 0$$

Or $$I = \frac{LC \pm \sqrt{(LC)^2 - 8}}{4}$$

For real I, $(LC)^2 \geq 8$.

If $(LC)^2 = 8$, a special case of it i.e. $H = 3I = \pm\sqrt{\frac{3}{2}}$ reduces the space into C^v-reducible Finsler space. Thus for $(LC)^2 > 8$ the space becomes semi C^v-reducible which is not a C^v-reducible Finsler space.

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ON NUMERICAL RANGE AND SPECTRUM OF LINEAR OPERATORS ON G.S.I.P. SPACES

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ABSTRACT

We prove some results on Numerical ranges and spectrum of bounded linear operators on generalised semi inner product spaces.

Keywords : Generalised semi-inner product, strictly convex, point spectrum, Numerical range.

INTRODUCTION

Definition : A Generalised semi-inner product space is a complex linear space X equipped with a generalised semi-inner-product (g.s.i.p.) $[.,.]$: Xx$X \to C$ satisfying the following conditions.

1. $[\alpha x+\beta y,z]=\alpha[x,z]+\beta[y,z]$
2. $[x,x]>0 \; for \; x\neq 0$
3. $|[x,y]|\leq[x,x]^{1/p}[y,y]^{p-1/p} \qquad 1<p<\infty$

for all x, y, z in X and for all α, β in C.

A g.s.i.p. space is a normed linear space whose norm is given by $\|x\|=[x,x]^{1/p}$. It is also proved every normed linear space can be made into a g.s.i.p. space [1].

Definition : A Homogeneous g.s.i.p. space X is one whose g.s.i.p. $[.,.]$ satisfies the additional condition that

$$[x,\alpha y]=\overline{\alpha}|\alpha|^{p-2}[x,y]$$

for all x, y in X and for all complex α.

Infact, every normed linear space X is a homogeneous g.s.i.p. space.

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For if $x \in X$, then by Hahn-Banach Theorem there is an x^* in X^* (normed conjugate of X) such that $x^*(x) = \|x\|^p$ and $\|x^*\| = \|x\|^{p-1}$ where $1 \leq p < \infty$. Hence, for any αx, there exists $(\alpha x)^*$ in X^* such that

$$(\alpha x)^*(\alpha x) = \|\alpha x\|^p$$

$$= |\alpha|^p \|x\|^p = \bar{\alpha}\, \alpha |\alpha|^{p-2} x^*(x).$$

$$= \bar{\alpha} |\alpha|^{p-2} x^*(ax).$$

Therefore, $(\alpha x)^* = \bar{\alpha}|\alpha|^{p-2} x^*$. Now, Let ϕ associate each x in X to exactly one such x^* in X^* and αx to $\bar{\alpha}|\alpha|^{p-2} x^*$ (for all complex α). We call $\phi : X \to X^*$ a generalised support mapping. It is easy to see that ϕ defines a homogeneous g.s.i.p. $[.,.]$ on X consistent with the norm of X if we set $\phi(x)(y) = [y, x]$ for all x, y in X. Since there are infinitely many generalized support mappings from X into X^*, we have an infinite number of homogeneous g.s.i.p's on X consistent with the norm of X. However, if the space X is smooth, there is a unique generalized support mapping ϕ, and hence there is a unique compatible homogeneous g.s.i.p. on X.

Theorem 1 : If X and Y are g.s.i.p. spaces, then the direct sum

$$X \oplus Y = \{(x, y) : x \in X, y \in Y\}$$

is a g.s.i.p. space with component wise addition and scalar multiplication together with the g.s.i.p. defined by

$$[(x_1, y_1), (x_2, y_2)] = [x_1, x_2] + [y_1, y_2].$$

The norm on $X \oplus Y$ is given by

$$\|(x, y)\| = \left(\|x\|^p + \|y\|^p\right)^{1/p} \qquad 1 < p < \infty$$

Proof : Clearly $X \oplus Y$ is a linear space. The linearity in the first argument

and the strict positiveness of [.,.] can be easily verified. Hence, we need to show Holder's inequality :

$$\left\|\left[(x_1,y_1),(x_2,y_2)\right]\right\| \le \left|\left[(x_1,y_1),(x_1,y_1)\right]^{1/p}\left[(x_2,y_2),(x_2,y_2)\right]^{p-1/p}\right|.$$

We have

$$\begin{aligned}\left\|\left[(x_1,y_1),(x_2,y_2)\right]\right\| &\le \left\|\left[(x_1,x_2)\right]\right\| + \left\|\left[(y_1,y_2)\right]\right\| \\ &\le \|x_1\|\,\|x_2\|^{p-1} + \|y_1\|\,\|y_2\|^{p-1} \\ &\le \left(\|x_1\|^p + \|y_1\|^p\right)^{1/p}\left(\|x_2\|^{(p-1)q} + \|y_2\|^{(p-1)q}\right)^{1/q} \qquad \left(1/p + 1/q = 1\right) \\ &= \left(\|x_1\|^p + \|y_1\|^p\right)^{1/p}\left(\|x_2\|^p + \|y_2\|^p\right)^{p-1/p} \\ &= \left([x_1,x_1] + [y_1,y_1]\right)^{1/p}\left([x_2,x_2] + [y_2,y_2]\right)^{p-1/p} \\ &= \left[(x_1,y_1),(x_1,y_1)\right]^{1/p}\left[(x_2,y_2),(x_2,y_2)\right]^{p-1/p}.\end{aligned}$$

Let A and B be bounded linear operators on g.s.i.p. spaces X and Y respectively, then the bounded linear operator $A \oplus B$ on $X \oplus Y$ is defined by

$$(A \oplus B)(x,y) = (Ax, By)$$

for all (x,y) in $X \oplus Y$.

Let X be a g.s.i.p. space and $T \in B(X)$. Then the Numerical range $W_p(T)$ is defined by $W_p(T) = \{[Tx,x] : \|x\| = 1\}$.

Theorem 2 : Let A and B bounded linear operators on homogeneous g.s.i.p. spaces X and Y respectively such that $W_p(A)$ and $W_p(B)$ are convex subsets of C. Then

$$W_p(A \oplus B) = co\left[W_p(A) \cup W_p(B)\right].$$

Proof : Let $\lambda \in W_p(A \oplus B)$ then we can find an element (x,y) in $X \oplus Y$ such that

$\|(x,y)\| = \left(\|x\|^p + \|y\|^p\right)^{1/p} = 1$ and

$\lambda = [(A \oplus B)(x,y),(x,y)] = [Ax,x] + [By,y]$.

Putting, $\|x\|^p = \alpha$, we see that $0 \le \alpha \le 1$ and $\|y\|^p = 1-\alpha$. Note that $\lambda \in W_p(B)$ when $\alpha = 0$ and $\lambda \in W_p(A)$ for $\alpha = 1$.

Let $0 < \alpha < 1$. Put $x' = \dfrac{x}{\alpha^{1/p}}$ and $y' = \dfrac{y}{(1-\alpha)^{1/p}}$.

Then x' and y' are the unit vectors in X and Y respectively.

Now, $\alpha[Ax',x'] + (1-\alpha)[By',y']$

$$= \alpha\left[\frac{Ax}{\alpha^{1/p}}, \frac{x}{\alpha^{1/p}}\right] + (1-\alpha)\left[\frac{By}{(1-\alpha)^{1/p}}, \frac{y}{(1-\alpha)^{1/p}}\right]$$

$$= \alpha.\frac{1}{\alpha^{2/p}}.\frac{1}{\alpha^{p-2/p}}[Ax,x] + (1-\alpha).\frac{1}{(1-\alpha)^{2/p}}.\frac{1}{(1-\alpha)^{p-2/p}}[By,y]$$

$$= \frac{\alpha}{\alpha}[Ax,x] + \frac{1-\alpha}{1-\alpha}[By,y]$$

$= [Ax,x] + [By,y] = \lambda.$

This shows that $\lambda \in co\left(W_p(A) \cup W_p(B)\right)$.

Conversely, suppose $\lambda \in co\left(W_p(A) \cup W_p(B)\right)$. Then $\lambda = \beta\mu + (1-\beta)\gamma$ with $0 \le \beta \le 1$, $\mu \in W_p(A)$ and $\gamma \in W_p(B)$.

There exist unit vectors x and y in X and Y respectively. Such that $\mu = [Ax,x]$ and $\gamma = [By,y]$.

Then $\lambda = \beta[Ax,x] + (1-\beta)[By,y]$

$$= \beta^{1/p}\beta^{1/p}\left(\beta^{1/p}\right)^{p-2}[Ax,x] + (1-\beta)^{1/p}(1-\beta)^{1/p}\left((1-\beta)^{1/p}\right)^{p-2}[By,y]$$

$$= \left[\beta^{1/p}Ax, \beta^{1/p}x\right] + \left[(1-\beta)^{1/p}By, (1-\beta)^{1/p}y\right]$$

$$= \left[A\left(\beta^{1/p}x\right), \left(\beta^{1/p}x\right)\right] + \left[B\left((1-\beta)^{1/p}y\right), \left((1-\beta)^{1/p}y\right)\right]$$

$$= \left[A\left(\beta^{1/p}x\right), B\left((1-\beta)^{1/p}y\right), \left(\beta^{1/p}x, (1-\beta)^{1/p}y\right)\right]$$

$$= \left[(A\oplus B)\left(\beta^{1/p}x, (1-\beta)^{1/p}y\right), \left(\beta^{1/p}x, (1-\beta)^{1/p}y\right)\right].$$

Since $\left\|\left(\beta^{1/p}x, (1-\beta)^{1/p}y\right)\right\| = 1$, it follows that $\lambda \in W_p(A\oplus B)$.

THEOREM 3 : Let X and Y be g.s.i.p. spaces. Suppose $A : X \to X$ and $B : Y \to Y$ are bounded linear operators. Then $A\oplus B$ bounded from below if and only if A and B are bounded from below.

Proof : Suppose $A\oplus B$ is bounded from below. Then there exists m>0 such that

$$\|(A\oplus B)(x,y)\| \geq m\|(x,y)\|$$

for all $(x,y) \in X\oplus Y$.

This gives $\|Ax\|^p + \|By\|^p \geq m^p\left(\|x\|^p + \|y\|^p\right)$ (*)

for all $x \in X$ and $y \in Y$. Taking $y=0$ in (*), we observe that $\|Ax\| \geq m\|x\|$ for all $x \in X$. Hence A is bounded from below.

Similarly B is also bounded from below.

Conversely,

Suppose A and B are bounded below. Then there exists positive constants α and β such that

$\|Ax\| \geq \alpha\|x\|$ for all $x \in X$.

$\|By\| \geq \beta\|y\|$ for all $y \in Y$.

If $m = \min(\alpha, \beta)$. Then it is trivially true that

$$\|(A \oplus B)(x,y)\| \geq m\|(x,y)\|$$

for all $(x,y) \in X \oplus Y$.

THEOREM 4 : Let T be a bounded linear operator on a strictly convex g.s.i.p. space X. If $\lambda \in W_p(T)$ with $|\lambda| = \|T\|$, then λ is in the point spectrum $\sigma_p(T)$.

Proof : Let x and y be non-zero vectors of x such that $[x,y] = \|x\| \, \|y\|^{p-1}$. Since X is strictly convex, by Lemma 3 ([2], p. 920) there exists a real number $\alpha > 0$ such that $y = \alpha x$.

Let $\lambda \in W_p(T)$ such that $|\lambda| = \|T\|$, then there exists an $x \in X$ such that $\|x\| = 1$ and $\lambda = [Tx, x]$.

Now, $\|T\| = |\lambda| = |[Tx,x]| \leq \|Tx\| \, \|x\|^{p-1} \leq \|T\|$.

This shows that

$$|[Tx,x]| = \|Tx\| \, \|x\|^{p-1}$$

Let θ be such that

$$e^{i\theta}[Tx,x] = |[Tx,x]|.$$

Then $[e^{i\theta}Tx, x] = \|e^{i\theta}Tx\| \, \|x\|^{p-1}$.

Since X is strictly convex, there exists $\alpha > 0$ such that $x = \alpha e^{i\theta}Tx$. Hence $Tx = \beta x$ where $\beta = \alpha^{-1}e^{-i\theta}$, a complex number. Now $\beta \in \sigma_p(\mathrm{T})$ and $\beta = \beta[x,x] = [\beta x, x] = [Tx,x] = \lambda$ and hence $\lambda \in \sigma_p(T)$.

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AIR POLLUTION SO_2 AFFECTING THE CLOUD MODIFICATION

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ABSTRACT

Lower troposphere is filled with trace gases as air pollutants. Sulphure dioxide (SO_2) is the main component. The critical radii, critical composition and nucleation rate of sulfuric acid droplets formed through binary homogeneous nucleation process under a variety of ambient conditions, are calculated. The main purpose of the present study is to calculate the pH values of $H_2SO_4.nH_2O$ droplets with different weight percentage of H_2SO_4 It is found that pH values are positive for weight percentage of acid less than 9.959%, but it is negative for weight percentage more than 9.959%.

Solar activity controls the formation of liquid aerosol ($H_2SO_4.nH_2O$) particles through the conversion of SO_2 into H_2SO_4 in presence of sunlight. It has been concluded that the values of pH decrease with increase in weight percentage of H_2SO_4. This may have significant implications on cloud structure and hence on the atmospheric climates.

Key Words and phrases : Air pollution, Weight percentage, Binary nucleation, pH value and Nucleation rate.

INTRODUCTION

It is well known that aerosol particles in the lower troposphere are formed through oxidation of SO_2 in the presence of water vapour. Two different processes involve in the formation of homogeneous and the heterogeneous nucleation. The ambient process of water vapour and gaseous sulfuric acid in troposphere helps in calculation of nucleation rates to estimate the physical and chemical properties.

The nucleation rates of $H_2SO_4.nH_2O$ aerosol nuclei have been computed [18] under different atmospheric conditions. The results of measurements of the concentration of gas phase H_2SO_4 performed during the pacific Exploratory mission have been presented [13]. During a night time portion of one flight, the

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H_2SO_4 was found to increase with decreasing relative humidity. The gas phase H_2SO_4 has atmospheric life time of the order of hour or less being lost by condensation, rainout or dry deposition. In areas away from the Earth's surface, condensation of H_2SO_4 is thought to occur predominately on the surface of pre-existing aerosol particles [1,3]. However, gas phase is thought to initiate new particle formation, if the existing particle surface area is small [10].

Growth of condensation nuclei composed of water soluble substances in the moist air and related problems of nucleation in binary system have been studied by several investigators [12, 15, 16, 20]. Gleitsmann and Zellner [7] investigated the formation of cloud condensation nuclei (C.C.N.) in the jet regime modeling studies. A model study for the formation of C.C.N. in jet regime air craft plumes has been proposed. Simple model is developed to describe the formation of particles from considerable vapours in different atmospheric circumstances [5]. A model has been presented to simulate heterogeneous reaction of nitrogen, chlorine and bromine compounds on and in sulfate aerosols under conditions encountered in lower moist stratosphere and under troposphere [9]. A model was applied to the daily average concentrations in available computer programme [17].

Sulfuric acid or other condensable in atmosphere will be transferred to aerosol particles or droplets at a rate depending on the vapour concentration. The mass and heat transfer process [2], involve macroscopic diffusion for large drops, and surface molecular process for small drops. Although here is much recent theory on subject [19], it has not yet been applied to sulfuric acid water condensation. It has been proposed that the value of molecular sticking probability for sulfuric acid molecules be unity [4].

The pH values of H_2SO_4.n H_2O clusters are also calculated which help to find out information about acidic rain fall. Rain water collected in cities which were free from pollution of industry, was alkaline in nature while near the industrial areas it was acidic. OECD project (1972-75) has shown that acidic rain fall can be possible in Europe. A downward trend in pH has been reported that the rain fall in Hawaii is acidic [14]. The sulphates and nitrates are responsible for acidic rain. pH values in rain water in India might be restricted to localized regions in highly industrialized cities like Bombay [11].

Measurement of pH with fluorescent have been undertaken in order to measure the pH evolution of acidic solutions under pressure up to 250 Mpa [8]. This technique is quite rapid and allows to monitor the changes of pH in real time. When pressure increases, pH is shown to change even for buffers. For example, distilled water (aqueous acid solution) with pH 5.8 has shown a decrease in pH by 0.30 and 0.31 when pressure increase to 100 Mpa and 200 Mpa, respectively. The decrease in pH at 100 Mpa and 200 Mpa, respectively. The decrease in pH at 100 Mpa has been shown earlier to be 0.73.

THEORETICAL CONSIDERATION

Heteromolecular Nucleation

Gas-to-particle conversion (SO_2- → H_2SO_4.n H_2O) process occurring in the lower atmosphere, is believed to be an important link in the troposphere. Several reactions have been proposed. It has been suggested that the first step in the formation of sulfate particles in the trimolecular process [6] :

$$SO_2 + O + M \xrightarrow[h\nu]{\text{sunlight}} SO_3 + M \tag{1}$$

where M is chemically neutral gas molecule. The oxygen atoms are generated by the photodissociation of O_2 molecules.

The reactions are followed by bimolecular collision, so that SO_3 quickly hydrates to sulfuric acid.

$$\underset{}{SO_3+H_2O} \rightarrow \underset{\text{(unstable)}}{SO_2(OH)_2} \rightarrow \underset{\text{(stable)}}{H_2SO_4} \tag{2}$$

In presence of excess water vapour, embryonic nuclei of sulfuric acid $H_2SO_4.nH_2O$ are formed through step by step addition of individual water molecules.

Nucleation theory of a Binary System

In gas-to-particle conversion, the phase transition involved may be heteromolecular homogenous or heterogeneous nucleation. The heterogeneous

nucleation is a process to form new particles with more than one substance. The particles in the troposphere and stratosphere are usually formed through the heteromolecular process. The liquid drop model has been considered. The formation and growth of spherical solution droplets through binary heteromolecular nucleation are controlled by Gibb's free energy change.

$$\Delta G\,(n_A,n_B) = n_A(\mu^l_A-\mu^g_A) + n_B(\mu^l_B-\mu^g_B) + 4\pi r^2\sigma \tag{3}$$

where n_i is the number of moles of species i (A and B); μ^l_i and μ^g_i are the chemical potential of species i; r, the radius of the embryo, and σ, the macroscopic surface tension. Subscript A refers to water and B refers to solute or the acid. The superscript 'l' denotes the liquid phase and 'g' the gas phase. Here it has been assumed to be spherical and the embryo is sufficiently large that the rotational and vibrational contribution to free energy can be completely neglected.

The change in chemical potential from gas phase to liquid phase is given by

$$\mu^l_i-\mu^g_i = RT\log_e(P_i/P_i^{sol}) \tag{4}$$

where R is the universal gas constant; T, the absolute temperature; p_i, the environment (ambient) vapour pressure of species 'i' and p_i^{sol}, the equilibrium partial vapour pressure of 'i' over a flat surface of the solution. The equation (3) may also be written as

$$\Delta G\,(n_A,\, n_B)=-n_A\,RT\,log_e\,(S_A/a_A)-n_B\,RT\,log_e(S_B/a_B)+4\pi r^2\sigma \tag{5}$$

where the quantities S_A, S_B, a_A and a_B are defined as

$S_A = P_A\,/\,P^O_A$ = Relative humidity

$S_B = P_B\,/\,P^O_B$ = Relative acidity

$a_A = P_A^{sol}\,/\,P^O_A$ = Water activity

$a_B = P_B^{sol}\,/\,P^O_B$ = Acid activity

where P^o_A and P^o_B are the equilibrium vapour pressure of A and B, respectively over a flat surface of pure substance.

The radius of embryo is given by

$$\left(\frac{4}{3}\right)\pi r^3 \rho = n_A M_A + n_B M_B \tag{6}$$

where ρ is the density of solution; M_A and M_B, molecular weight of A and B, respectively.

Equilibrium Size of nucleus

At equilibrium, the saddle point is reached, which may be determined using the critical conditions

$$\left(\frac{\partial \Delta G}{\partial n_A}\right)_{n_B} = 0 \tag{7}$$

and,
$$\left(\frac{\partial \Delta G}{\partial n_B}\right)_{n_A} = 0 \tag{8}$$

Solving eqn. (7) and eqn. (8), the critical radius of nucleus is given by

$$r^* = [3(n_A M_A + n_B M_B)/4\pi\rho N_0]^{1/3} \tag{9}$$

N_0, being Avogadro's number

pH value of H_2SO_4.n H_2O Clusters

The pH value of $H_2SO_4.nH_2O$ clusters is given by

$$\text{pH} = \log_{10} \{[(C)^{\log_{10}(N'/N)}][H^+]_N\}^{-1} \tag{10}$$

where $C = 7.9432$, known as normal ratio constant (say); $[H^+]_{N'}$, the hydrogen ion concentration in known amount of normal solution; N, normality of known solution and N', the normality of unknown solution.

RESULTS AND DISCUSSION

The critical radius of water droplet is obtained by equation (9). From equation it is evident that radius is the function of n_A and n_B. Hence the radius is also the function of weight percentage of sulfuric acid. For different weight percentage of sulfuric acid, the hydrogen ion concentration $[H^+]_{N'}$ and normality N' of sulfuric acid solution also changes. We have calculated the values of $[H^+]_{N'}$,

N' and pH for different weight percentage of sulfuric acid. The calculated values are shown in the following table.

Table 1 :- The calculated values of r^*, N', $[H^+]_{N'}$ and pH as the function of n_A and weight percentage for given value of n_B=1

n_A	n_B	Wt% of H_2SO_4	r^* (A^0)	N'(g/L)	$[H^+]_{N'}$	pH
539	1	1.000	15.69	0.205	0.120	0.920
176	1	3.000	10.83	0.624	0.328	0.484
104	1	5.000	9.11	1.077	0.523	0.282
72	1	7.000	8.09	1.499	0.722	0.142
55	1	9.000	7.41	1.947	0.913	0.034
52	1	9.478	7.28	2.056	0.959	0.018
51	1	9.646	7.23	2.095	0.975	0.011
50	1	9.820	7.19	2.134	0.992	0.004
49.22+	1	9.959	7.17	2.155	1.000	0.000
49	1	10.000	7.14	2.176	1.009	-0.004
48	1	10.187	7.09	2.222	1.028	-0.012
47	1	10.381	7.046	2.266	1.047	-0.019
44	1	11.000	6.900	2.411	1.107	-0.044

\+ critical value which corresponds to zero pH.

From the above table it is evident that the pH values for clusters of H_2SO_4. nH_2O are positive for which weight percentage of H_2SO_4 in the cluster is less than 9.959% while pH values are negative for the cluster with weight percentage of H_2SO_4 in the cluster more than 9.959%. This is called critical weight percent of H_2SO_4 for given n_A and n_B. Also the pH value of clusters are found to be inversely proportional to the weight percentage of sulfuric acid available in the cluster. The hydrogen ion concentration and normality of clusters are directly proportional to the weight percentage of sulfuric acid.

The radius of critical sized cluster (or nucleus) decreases with number of constituent water molecules. Thus, we find the pH values decrease with decrease in values of n_A, the number of water molecules in the cluster, while hydrogen ion concentration shows increasing trend with decrease in n_A.

CONCLUSION

From above it may be concluded that in areas of highly polluted atmosphere, the acidity increases in rain water with increase in weight percentage of $H_2SO_4.nH_2O$ clusters. The nucleation rate i.e. number of critical sized nuclei also increases but at the same time radius of nuclei decreases. Hence small nuclei are formed, which further grow in droplet following the condensation of water vapours.

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A NOTE ON CHARACTERISATIONS OF STRICT CONVEXITY

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ABSTRACT

We give proofs of some Characterisations of strict convexity for normed linear spaces.

Key words : Strict convexity, Normed linear space, Semi-inner-product space, Duality mapping.

2000 Mathematics Subject Classification: 46C15, 46C 50

INTRODUCTION

We say that φ is a generalised duality mapping of a Banach space X into its dual X^* if to each x in X it assigns a unique element in the set $\varphi(x)$ in X^* determined by

$$j(x)=\left\{x^*\in X^*: x^*(x)=\left\|x\right\|^p \, and \left\|x^*\right\|=\left\|x\right\|^{p-1}, 1\le p<\infty\right\}.$$

We note that $\varphi(x)$ is a non-empty set by Hahn-Banach theorem. There are infinite numbers of such mappings unless the space is smooth. Now if we set $\varphi(x)(y)=[y,x]$, then it can be seen that [.,.] is linear in the first argument, strictly positive and satisfies the Holder's inequality. The Mapping [.,.] will be called the generalised semi-inner-product (g.s.i.p) determined by φ. Clearly each φ defines a g.s.i.p compatible with the norm of X [4]. We observe that for any (αx) there exists $(\alpha x)^*$ in X^* such that

$$(ax)^*(ax)=\left\|ax\right\|^p=\left|a\right|^p\left\|x\right\|^p=\bar{a}\left|a\right|^{p-2}x^*(ax).$$

Now let φ associate αx to $\bar{a}\left|a\right|^{p-2}x^*$. Then it follows that

$$[x,ay]=\bar{a}\left|a\right|^{p-2}[x,y] \qquad \forall x,y\in X.$$

For $p=2$, we get an s.i.p of Lumer [3].

* Department [illegible]

A Normed linear space is said to be Strictly convex if and only if $\|y+z\|=\|y\|$ and $[z,y]=0$ imply $z=0$, where [.,.] is any g.s.i.p consistent with the norm of X. This is true in a s.i.p space [5]. Strict convexity has proved to be useful in studies of the Geometry of Banach spaces [1]. In this paper we give proofs of some characterizations of strict convexity for normed linear spaces.

Theorem 1: Let X be a normed linear space and [.,.] be any g.s.i.p compatible with the norm of X. If $x,y \in X$ such that $\|x+y\|=\|x\|+\|y\|$, then $[x, x+y] = \|x\| \, \|x+y\|^{p-1}$ and $[y,x+y] = \|y\| \, \|x+y\|^{p-1}$ $(1<p<\infty)$.

Proof: we have $\|x+y\|^p=[x+y,x+y]=[x,x+y]+[y,x+y]=\|x+y\|^{p-1}(\|x\|+\|y\|)$

Hence

(a) $\{\|x\| \, \|x+y\|^{p-1}-Re([x,x+y])\}+\{\|y\| \, \|x+y\|^{p-1}-Re([y,x+y])\}=0.$

But we have

(b) Re $([x,x+y])\leq|[x,x+y]|\leq\|x\| \, \|x+y\|^{p-1}$.

(c) Re $([y,x+y])\leq|[y,x+y]|\leq\|y\| \, \|x+y\|^{p-1}$.

Since the inequalities (b) and (c) imply that each term in (a) is non-negative, we have Re $\{[x,x+y]\}=\|x\| \, \|x+y\|^{p-1}$

and

$$\text{Re } \{[y,x+y]\}=\|y\| \, \|x+y\|^{p-1}.$$

Now these equalities together with (b) and (c) imply the desired results.

Theorem 2: Let X be a normed linear space and [.,.] be any g.s.i.p consistent with the norm of X. Then $\|y+z\|=\|y\|$ and $[z,y]=0$ imply $z=0$ if and only if $x\neq y$ and

$\|x\| = \|y\| = 1$ imply $\|(\frac{1}{2})(x+y)\| < 1$.

Proof: Suppose $\|y+z\| = \|y\|$ and $[z, y] = 0$ together imply $z = 0$.

Let $x \neq y$ and $\|x\| = \|y\| = 1$. Then $\|(\frac{1}{2})(x+y)\| \leq 1$ Suppose that $\|(\frac{1}{2})(x+y)\| = 1$. Then we have $\|x+y\| = \|x\| + \|y\|$. Now by Theorem 1 it follows that $[x, x+y] = [y, x+y]$. Hence $[x - y, x + y] = 0$. Also we observe that $\|(x+y)+(x-y)\| = \|x+y\|$. Hence by the assumption $x - y = 0$. Thus $x = y$. This contradiction shows that $\|(\frac{1}{2})(x + y)\| < 1$. Conversely, Let $x \neq y$ and $\|x\| = \|y\| = 1$ imply $\|(\frac{1}{2})(x + y)\| < 1$. Suppose that $x^*(x) = \|x^*\| = x^*(y)$, Where x^* is a non-zero element in X^*. Let $z = (\frac{1}{2})(x+y)$. Then $\|x^*\| = \frac{1}{2}\{x^*(x) + x^*(y)\} = x^*(z)$. Hence $\|x^*\| = |x^*(z)| \leq \|x^*\| \|z\|$. This shows that $1 \leq \|z\| = \|(\frac{1}{2})(x+y)\| \leq \frac{1}{2}(\|x\| + \|y\|) = 1$. Hence $\|(\frac{1}{2})(x + y)\| = 1$. So by the hypothesis, it follows that $x = y$. Hence we have shown that each non-zero x^* in X^* attains its maximum on atmost one point of the unit sphere in X. We now show $\|y + z\| = \|y\|$ and $[z, y] = 0$ imply $z = 0$. Let us suppose that this not true. Then there exists $y, z \neq 0$ in X such that $\|y + z\| = \|y\|$ and $[z, y] = 0$. Now it can be easily shown that the function $f_y: X \to C$ defined by $f_y(x) = [x, y]$ belongs to X^*. Hence $f_y(y + z) = [y + z, y] = [y, y] = f_y(y)$. This shows that $f_y(y)$ $\|y\|^p = \|y\|^{p-1}\|y\| = \|f_y\| \|y\| = \|f_y\| \|y+z\|$. So f_y attains its maximum at two points y and $y + z$. This contradiction proves the result.

Theorem 3: Every non-zero x^* in X^* attains a maximum on atmost one point of the unit sphere in X if and only if $\varphi(x) \cap \varphi(y) = \phi$ whenever $x \neq y$.

Proof: Suppose the necessary condition holds and $x \neq y$. If $\varphi(x) \cap \varphi(y) \neq \phi$, then there exists an $f \in X$ such that $f \in \varphi(y)$. From this we have $f(x) = \|f\| \|x\|, \|f\| = \|x\|^{p-1}$ and $f(y) = \|f\| \|y\|, \|f\| = \|y\|^{p-1}$.

Hence $\|x\| = \|y\|$, $f(x) = \|f\| \|y\|$ *and* $f(y) = \|f\| \|y\|$. These equalities imply that f attains

maximum at two points which is a contradiction, and hence $\varphi(x) \cap \varphi(y) = \phi$

Conversely, Let $x \neq y$ imply $\varphi(x) \cap \varphi(y) = \phi$. Suppose, there exists a functional $f \in X^*$ which attains its maximum at two distinct points x and y. Then $f(x) = \|f\| \|x\|$ and $f(y) = \|f\| \|y\|$. This shows that $f \in \varphi(x)$ and $f \in \varphi(y)$. From this we have $x \neq y$ implies $\varphi(x) \cap \varphi(y) \neq \phi$. Which contradicts our hypothesis.

Note: For $p = 2$, this reduces to Theorem *4* of Guddar and Strawther [2].

Theorem 4: If x and y are two distinct unit vectors such that $\|(x+y)/2\| < 1$, then $\|tx+(1-t)y\| < 1$. $(0 < t < 1)$

Proof: Let $z = (\frac{1}{2})(x + y)$. If $0 < t < \frac{1}{2}$ and $s = 2t$, then $o < s < 1$. Now $s z + (1-s) y = (\frac{1}{2}) sx + ((\frac{1}{2})s + 1 - s) y = t x + (1-t)y$. By hypothesis, $\|z\| < 1$ and hence

$$\|tx+(1-t)y\| = \|sz\| + \|(1-s)y\| = s\|z\| + (1-s)\|y\| < 1.$$

If $\frac{1}{2} < t < 1$ and $s = 2(1-t)$, then $0 < s < 1$. Now $s z + (1-s) x = ((\frac{1}{2}) s + 1-s) x + (\frac{1}{2}) sy = t x + (1-t) y$. Again by the hypothesis $\|z\| < 1$ and hence

$$\|tx+(1-t)y\| = \|sz\| + \|(1-s)x\| = s\|z\| + (1-s)\|x\| < 1.$$

Theorem 5: Let X be a normed linear space and $x, y \in X$. Then $\|x+y\| = \|x\| + \|y\|$ imply $y = \alpha x$ $(\alpha > 0)$ if and only if $x \neq y$ and $\|x\| = 1 = \|y\|$ imply $\|(\frac{1}{2})(x+y)\| < 1$.

Proof: Suppose that $\|x+y\| = \|x\| + \|y\|$ implies $y = \alpha x$ for some $\alpha > 0$. We show that $x \neq y \in X$ and $\|x\| = 1 = \|y\| \Rightarrow \|(\frac{1}{2})(x+y)\| < 1$. On the contrary suppose x, y be such that $\|x\| = \|y\| = 1$ *and* $\|(\frac{1}{2})(x+y)\| = 1$. Then $\|x+y\| = 2 = \|x\| + \|y\|$. Hence by assumption $y = \alpha x$ for some $\alpha > 0$. Hence $\|y\| = \|ax\| = a\|x\|$ *so* $a = 1$. Thus $x = y$.

Conversely, *let* $x \neq y$ be such that $\|x\| = 1 = \|y\| \Rightarrow \|(\frac{1}{2})(x+y)\| < 1$. Let x and y be nonzero elements in X such that $\|x+y\| = \|x\| + \|y\|$. Put $\lambda = \|x\|, m = \|y\|$, $x_1 = \frac{x}{\lambda}$ *and* $y_1 = \frac{y}{m}$ Then

$\|x_1\| = 1 = \|y_1\|$ and $\|\lambda x_1 + \mu y_1\| = \|x + y\| = \|x\| + \|y\| = 1 + \mu$. Now put $t = \lambda/(\lambda + \mu)$ and observe that $\|t\, x_1 + (1-t)y_1\| = 1$. Since $0 < t < 1$, by Theorem 4, it follows that $x_1 = y_1$. Hence $y = \alpha x$ with $\alpha = \mu / \lambda$.

Theorem 6: Let X be a normed linear space and [.,.] be any g.s.i.p. compatible with the norm of the space. Then $\|x + y\| = \|x\| + \|y\| \Rightarrow y = \alpha x$ for some $\alpha > 0$ if and only if $\|y + z\| = \|y\|$ *and* $[z, y] = 0 \Rightarrow z = 0$.

This theorem follows from Theorem 2 and Theorem 5.

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STABILITY OF A SOCIAL GROUP

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ABSTRACT

We have studied the stability behaviour of a Social Group (a group of persons developed by Simon [7], drawing upon Homans' theoretical frame, by means of loop analysis based on Levin's formulation [3] and thermodynamic criteria of stability. The stochastic criteria of stability of this system has also been presented here.

Key words : Social Group, Stability, Loop analysis, Thermodynamic Model, Stochastic Model.

AMS (MOS) subject classification (1980) : 92A17

INTRODUCTION

We consider a system of social group which satisfies the following postulates [7] :

(a) Interaction intensity is due to the combined effects of two communication causes, i.e. by friendliness and by activity.

(b) If a group characterized by little initial friendliness is induced to interact, friendship will increase, but there will be a lag.

(c) Activity rate accomodates itself to the level of friendliness and to the activity imposed externally.

Here we have studied the stability of equilibrium of this system by means of loop analysis based on Levins's formulation [3] in which the interactions between species can be specified in a qualitative but not a quantitative way. The present paper also consist of the thermodynamic and stochastic modelling of the system and study of the stability of equilibrium from the consideration of thermodynamic and stochastic criteria of stability.

MATHEMATICAL EQUATIONS : STABILITY OF EQUILIBRIUM STATE

Let $I(t)$, $F(t)$ and $E(t)$ denote the intensity of interaction, the level of friendliness among the members, the amount of activity carried on by members of the group and the amount of activity imposed by external influences respectively. We consider a social group (a group of persons) whose behaviour can be characterized

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by these four variables, all functions of time t. Then the mathematical model of this system of social group developed by Simon [7] is

$$I = a_1 F + a_2 A \qquad (1a)$$

$$\frac{dF}{dt} = b\,(I - \beta F) \qquad (1c)$$

$$\frac{dA}{dt} = c_1\,(F - \gamma A) + c_2\,(E - A) \qquad (1c)$$

where a_1, a_2, b, β, c_1, c_2 and γ are coefficients which are assumed to be positive constants. Here $I(t)$ and $A(t)$ are encogeneous (dependent) variables whose values are determined within the system,, while $E(t)$ is an exogeneous (independent) variable.

From equations (1) we have

$$\frac{dF}{dt} = a_{11} F + a_{12} A \qquad (2a)$$

$$\frac{dA}{dt} = a_{21} F + a_{22} A + c_2 E \qquad (2b)$$

where $a_{11} = -b\,(\beta - a_1)$, $a_{12} = ba_2$, $a_{21} = c_1$ *and* $a_{22} = -(c_1\gamma + c_2)$ (2c)

Here we assume that E is a positive constant. The system (2) has equilibrium (stationary) solution

$$F_e = \frac{a_2 c_2 E}{(\beta - a_1)(c_1 r - c_2) - a_2 c_1} \qquad (3)$$

$$A_e = \frac{(\beta - a_1) c_2 E}{(\beta - a_1)(c_1\gamma + c_2) - a_2 c_1}, \qquad (4)$$

provided $\beta > a_1 + \dfrac{a_2 c_1}{(c_1\gamma + c_2)}$ (5)

Let $F = F_e + x_1$, $A = A_e + x_2$, the equations (2) give

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \qquad (6a)$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 \qquad (6b)$$

The eigen-value equation for the system (6) is

$$\lambda^2-(a_{11}+a_{22})\,\lambda+(a_{11}a_{22}-a_{12}a_{21})=0.$$

Therefore the eigen-value are

$$\lambda_1,\lambda_2=\frac{-\Lambda\pm\sqrt{\Delta}}{2} \tag{7}$$

where $\Lambda=-(a_{11}+a_{22})=b(\beta-a_1)+(c_1\gamma+c_2)>0$ [from(5)]

and $\Delta=(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})$

$$=\{b(\beta-a_1)+c_1\gamma+c_2)\}\_4b(c_1\gamma+c_2)\{\beta-(a_1+\frac{a_2c_2}{c_1\gamma+c_2})\}$$

$$<\Lambda^2 \qquad [\text{from}(5)]$$

$\therefore \lambda_1,\ \lambda_2<0$

Therefore, if the equilibrium state of the system (1) exists then it is asymptotically stable. Now (5) implies that

$$\beta F_e>a_1F_e \tag{8}$$

What this means is that the quantity of interaction (βF_e) necessary to produce the equilibrium level of frendliness must be greater than the amount of communication ($a_1\ F_e$) resulting from this equilibrium degree of friendliness, if equilibrium state exists which is also asymptotically stable.

This is a valuable conclusinon derived mathematically and easily tested empirically [4].

If E is a negative constant then the system (2) has equilibrium solution (3) & (4) provided $a_1<\beta<a_1+\frac{a_1c_1}{c_1\gamma+c_2}$. In this case one of the eigen values λ_1,λ_2 is positive and hence the system (1) is unstable.

STABILITY OF EQUILIBRIUM STATE : METHOD OF LOOP ANALYSIS

Let $X_1=F, X_2=A$, then we represent the system (2) by a diagram, based on Levin's formulation [3], in which each variable is represented by a point or vertex and the

relations amongst variables appear as oriented links connecting the vertices so that the line connecting X_j to X_i represents the interaction or effects of X_j on X_i and corresponds to the element a_{ij}. A link or series of links that leaves and eventually reenters the same vertex is called a loop. Corresponding to the diagonal matrix elements a_{ij} there are loops that connect each X_i to itself, termed self-loops. When two loops share no vertices in common, we refer to the loops as disjunct otherwise these are called conjunct. The feedback' at level K in a system is defined by

$$F_k = \sum_m (-1)^{m+1} \; L(m,k) \tag{9}$$

where $L(m,k)$ is the product of K vertices (i.e., coefficients a_{ij}) which from m disjunct loops: the summation is taken over all m and all posible products involving m loops.

The system (2) will be stable, i.e., it returns to its equilibrium state after a perturbation, if and only if [3]

$$F_K < 0 \; (K = 1, 2)$$

Here $a_{11} = - b\,(\beta - a_1) < 0,\ a_{12} = ba_2 > 0,\ a_{21} = c_1 > 0$ *and* $a_{22} = -(c_1\,\gamma + c_2) < 0.$

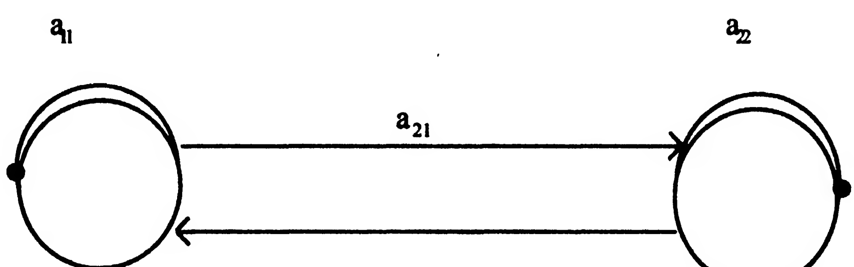

Fig.1

Figure 1 represents this case. Here friendliness promotes activity rate (since $a_{21}>0$) and is symbolized by an arrow from X_1 to X_2 and conversely activity rate promotes friendliness (since $a_{12} > 0$) and is symbolized by an arrow from X_2 to X_1. The growths of friendliness and activity rate are self-damped (since $a_{11}, a_{22} < 0$) and are expressed by the self-loops with circles on X_1, X_2.

Now, $$F_1 = \sum_i a_{ij} = -\{b(\beta - a) + (c_1\gamma + c_2)\} < 0 \tag{11}$$

$$F_2 = \sum_{i<j} a_{ij}\, a_{ji} - \sum_{i<j} a_{ii} a_{jj}$$

$$= a_{12}\, a_{21} - a_{11}\, a_{22}$$

$$= -b(c_1\gamma + c_2)\ \{\beta - (a_1 + \frac{a_2 c_1}{c_1\gamma + c_2})\} < 0$$

(since $\beta > a_1 + \dfrac{a_1 c_1}{c_1\gamma + c_2}$ for existence of equilibrium state).

Therefore, this system is stable.

This section presents a new method of analysis of stability behaviour of a system. If the interaction of a system of social group can be specified in a qualitative but not a quantitative way, then this formalism proves particularly useful for examining the properties of social groups.

THERMODYNAMIC MODEL AND STABILITY

The social group system can be considered as a net-work of flows of energy and biomass. An appropriate modelling of such a system can be carried out by the thermodynamics of irreversible processes. To develop a thermodynamic model of the system governed by the equations (6), we have to choose the thermodynamic fluxes j_i and forces F_i as

$$J_i = \frac{d\, x_i}{d\, t},\ F_i = x_i\ (i = 1, 2) \qquad (13)$$

The choice (13) is appropriate in the sense that in the equilibrium state (F_e, A_e) the thermodynamic forces F_i which are deviation from the equilibrium state (F_e, A_e) vanish and correspondingly the fluxes or flows also stop. With these choices we can thus write the system of equations (6) as

$$J_i = a_{11}\ F_1 + a_{12} F_2$$

$$J_2 = a_{21}\ F_1 + a_{22}\ F_2$$

which are the linear phenomenological relations of irreversible thermodynamics. If S is the entropy of the system for the non-equilibrium state near the equilibrium state, it can be expanded about the equilibrium state as

$$S = S_{eq} + (\delta S)_{eq} + \frac{1}{2} (\delta^2 S)_{eq} + \ldots \qquad (15)$$

Then the thermodynamic criteria of stability of the equilibrium state is given by [2, 1]:

$$\frac{d}{dt}\left(\frac{1}{2}\delta^2 S\right)_{eq} = -\sum_{1} \delta F_i \, \delta j_i > 0. \tag{16}$$

For the linear relations (14), this becomes

$$a_{11}\,(\delta F_1)^2 + (a_{12} + a_{21})\,\delta F_1\,\delta F_2 + a_{22}\,(\delta F_2)^2 < 0. \tag{17}$$

Implying negative definiteness of the matrix (a_{ij}). So the criteria of thermodynamic stability of the equilibrium state becomes

$$a_{11} < 0,\ a_{22} < 0,\quad 4(a_{11}\,a_{22}) > (a_{12} + a_{21})^2 \tag{18}$$

Now, first two conditions are satisfied since $\beta > a_1 + \dfrac{a_2\,c_1}{c_1\,\gamma + c_2}$ for existence of equilibrium state. The third condition gives

$$\beta > a_1 + \frac{(c_2\,a_1\,b)^2}{4b(c_1\,\gamma + c_2)} \tag{19}$$

$$\text{Now, } a_2\,c_1 + \frac{(c_1 + a_2\,b)^2}{4b} = -\frac{(c_1 - a_2\,b)^2}{4b} \le 0 \tag{20}$$

Therefore, the thermodynamic criteria of stability implies the existence and stability (deterministic) of equilibrium state but the converse is not true in general, If $c_1 = a_2\,b$, *i.e.*, $\dfrac{\partial}{\partial a}\left(\dfrac{dF}{dt}\right) = \dfrac{\partial}{\partial F}\left(\dfrac{dA}{dt}\right)$ *i.e.*, if the rate of change with respect to activity of the rate of change of friendliness with respect to time is equal to the rate of change with respect to friendliness of the rate of change of activity with respect to time, then the existence of equilibrium state implies thermodynamic stability and deterministic stability of this system.

STOCHASTIC MODEL AND STABILITY

The stochastic extension of the system of equations (92) is given by the system of stochastic differential equations

$$\frac{dF}{dt} = a_{11}\,F + a_{12}\,A + \Gamma_1\,(t) \tag{21a}$$

$$\frac{dA}{dt} = a_{21}\,F + a_{22}\,A + c_2 E + \Gamma_2\,(t) \tag{21b}$$

Where $\Gamma_i\ (t)_i = 1, 2)$ are the random perturbation terms which are due to the overall effect of the internal fluctuation between the level of friendliness among the members and the amount of activity carried on by members of the group.

These random perturbation terms Γ_i (t) are given by

$$< \Gamma_i(t) > = 0, \ <\Gamma_i(t)\ \Gamma_j\ (t')> = D_{ij}\ \delta(t-t')$$

$$D_{ij} = D\delta_{ij}\ (i, j = 1,2)$$

Where <.> represents the average over the ensemble of the stochastic process, D is the intensity of the noises, δ_{ij} is Kronecker delta function and $\delta(t)$ denotes the Dirac delta function.

The fluctuatuion intensities (variances) in F and A at any arbitrary instant t staisfying the stochastic differential equations (Eqs. 21 a and 21b) are given by [6] :

$$\sigma_F^2(t) = \int_0^t \left[\sum_{K=1}^{2} \sum_{s=1}^{2} G_{1K}(t')\ G_{1s}\ (t')\ D_{Ks} \right] dt' \tag{23a}$$

and

$$\sigma_A^2(t) = \int_0^t \left[\sum_{K=1}^{2} \sum_{s=1}^{2} G_{2K}(t')\ G_{2s}\ (t')\ D_{Ks} \right] dt' \tag{23b}$$

where

$$G_{ij} - \sum_{K=1}^{2} a_{iK}\, G_{kj} = 0, (i, j = 1, 2) \tag{24}$$

After some simplifications we have :

$$\sigma_F^2(t) = \frac{D}{2} \left[\begin{array}{c} \frac{1}{\lambda_1}\left(c_1'^2 + c_2'^2\right)\left(e^{2\lambda_1 t} - 1\right) + \frac{1}{\lambda_2}\left(c_1^{2''} + c_2^{2''}\right)\left(e^{2\lambda_1 t} - 1\right) \\ + \frac{4}{(\lambda_1 + \lambda_2)}\left(c_1'c_1'' + c_2'c_2''\right)\left\{e^{(\lambda_1+\lambda_2)t} - 1\right\} \end{array} \right] \tag{25a}$$

and

$$\sigma_A^2(t) = \frac{D}{2} \left[\begin{array}{c} \frac{1}{\lambda_1}\left(c_3'^2 + c_4'^2\right)\left(e^{2\lambda_1 t} - 1\right) + \frac{1}{\lambda_2}\left(c_3^{2''} + c_4^{2''}\right)\left(e^{2\lambda_2 t} - 1\right) \\ + \frac{4}{(\lambda_1 + \lambda_2)}\left(c_3'c_3'' + c_4'c_4''\right)\left\{e^{(\lambda_1+\lambda_2)t} - 1\right\} \end{array} \right] \tag{25b}$$

where c'_i . c''_i are arbitrary constants and λ_i (< 0) are given by (7). Therefore, σ_F^2 (t), σ_A^2 (t) converge exponentially of the limiting variances

$$\sigma_F^2(\infty) = \frac{D}{2}\left[\begin{array}{l}\left(\frac{|c_1'|}{\sqrt{n_1}} - \frac{|c_1''|}{\sqrt{n_2}}\right)^2 + \left(\frac{|c_2'|}{\sqrt{n_1}} - \frac{|c_2''|}{\sqrt{n_2}}\right)^2 \\ + \frac{\left(\sqrt{n_1} \pm \sqrt{n_2}\right)^2}{(n_1 + n_2)\sqrt{n_1 n_2}}\left(|c_1'c_1''| + |c_2'c_2''|\right)\end{array}\right] \tag{26a}$$

and

$$\sigma_A^2(\infty) = \frac{D}{2}\left[\begin{array}{l}\left(\frac{|c_3'|}{\sqrt{n_1}} - \frac{|c_3''|}{\sqrt{n_2}}\right)^2 + \left(\frac{|c_4'|}{\sqrt{n_1}} - \frac{|c_4''|}{\sqrt{n_2}}\right)^2 \\ +2\frac{\left(\sqrt{n_1} \pm \sqrt{n_2}\right)^2}{(n_1 + n_2)\sqrt{n_1 n_2}}\left(|c_3'c_3''| + |c_4'c_4''|\right)\end{array}\right] \tag{26b}$$

where $n_i = -\ \lambda_i (> 0)$.

Now, in an unvarying environment, the eigen values of the 2 x 2 interaction matrix of the system (6) are λ_1, λ_2 $= \frac{-\Lambda \pm \sqrt{\Delta}}{2}$ (from [7} . Therefore, the stability determining quantity is Λ and the deterministic stability criterion is satisfied, since $\Lambda > 0$. In a stochastic environment, whose random fluctuations have variance D, the stability provided by the interaction dynamics is again characterized by Λ. It is no longer enough that $\Lambda > 0$, for if $D >> \Lambda$, the system exhibits large fluctuatuins, which rapidly lead to extinction. In the intermediate region where Λ and D are commensurate, the system is likely to undergo significant fluctuations, even though they persist for long times. If $D << \Delta$, the fluctuatuins become very small, therefore the deterministic solution is recovered. These results are in good agreement with those of May [5].

CONCLUSIONS

We have studied the stability behaviour of the equilibrium state (provided it exists) of a social group developed by Simon [7], drawing upon Homans' theoretical frame, by means of loop anslysis based on Levins's formulation [3]and from the consideration of thermodynamic and stochastic criteria of stability. We have seen that the existence of equilibrium state enssumes the asymptotic stability of equilibrium state whereas the thermodynamic model needs different condiiton for stability. The thermodynamic criteria of stability ensures the existence and stability (deterministic) of equilibrium state but the converse is not true in general. The stochastic model not only depend on the parameters of the system but also on the fluctuatuation.

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A FIXED POINT THEOREM IN A 2- METRIC SPACE AND AN APPLICATION

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ABSTRACT

In this paper a class of self mappings of 2-metric spaces, which satisfy some nonexpansive type condition are considered and a fixed point theorem is established. As an application, a fixed point result on a product space is also obtained. Further, we show that Hsiao's remarks [2] on contractive maps in 2-metric spaces are incorrect.

MSC (2000) : 54 H 25.
Key Words : Fixed point, 2-metric space.

INTRODUCTION

The concept of 2-metric space has been investigated by Gähler in a series of papers [6]-[8]. A number of authors have studied the contractive and contraction type mappings in 2-metric spaces. For example see *K. Is'eki*. [9,10,11], S. L. Singh. [15,16,17]. Other related works by S.L. Singh. and others are found in [3], [4] and [9]-[17].

Gähler defined 2- metric space as follows:

A 2-metric on a set X with at least three points is a non-negative real-valued mapping d : $X \times X \times X \longrightarrow R$ satisfying the following properties:

(i) To each pair of points a, b with $a \neq b$ in X there in a point $c \in X$ such that $d(a,b,c) \neq 0$;

(ii) $d(a,b,c) = 0$ if at least two of the points are equal;

(iii) $d(a,b,c) = d(b,c,a) = d(a,c,b)$;

(iv) $d(a,b,c) \leq d(a,b,e) + d(a,e,c) + d(e,b,c)$ for all $a,b,c,e \in X$.

The pair (X, d) is called a 2-metric space. A sequence $\{x_n\}$ in X is called Cauchy

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if $\lim d(x_n, x_m, u) = 0$ for all $u \in X$. The sequence $\{x_n\}$ is convergent to $x \in X$ and x is the limit of this sequence if $\lim d(x_n, x, u) = 0$ for each $u \in X$. A 2-metric space in which every Cauchy sequence is convergent is called complete.

Here our aim is to obtain a fixed point theorem for a nonexpansive type map on a 2-metric space. An application of the same is also given. The nonexpansive type definition used herein is an extension of that of Ćirić [5] (see also [1]). For nonexpansive maps on 2-normed spaces, refer to Iséki [9].

MAIN RESULTS

Let (X, d) be a 2-metric space and $T : X \to X$ satisfying the following nonexpansive type conditions:

$$d(Tx, Ty, u)$$

$$\leq a.\max\{d(x,y,u),\ d(x,Tx,u),\ d(y,Ty,u),\ (\tfrac{1}{2})[d(x,Ty,u)+d(y,Tx,u)]\}$$

$$+b.\ max\{d(x,Tx,u),\ d(y,Ty,u)\}+c[d(x,Ty,u)+d(y,Tx,u)] \qquad (1)$$

for all $x,y,u \in X$, where a, b, c are non-negative real numbers such that

$$a+b+2c=1. \qquad (2)$$

First we study some properties of mapping T which satisfies (1) with a, b, c greater than or equal to zero.

Lemma 1. Let $T : X \to X$ be a mapping satisfying (1) where a, b, c satisfy (2) with $c > 0$ and x_0 be any point in X, then

$$d(x_n, x_{n+1}, x_{n+2})=0\ \forall n / \geq 0 \qquad (3)$$

where $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$

Proof. Since $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$ then by (1) we have

$$d(Tx_n, Tx_{n+1}, u)$$

$$\leq a.\max\{d(x_n,Tx_n,u),\ d(x_{n+1},Tx_{n+1},u),\ \tfrac{1}{2}d(x_n,Tx_{n+1},u)\}$$

$$+b.\ max\{d(x_n,Tx_n,u),\ d(x_{n+1},Tx_{n+1},u)\}+c.d(x_n,Tx_{n+1},u) \qquad (4)$$

for all $u \in X$.

Thus $d(Tx_n, Tx_{n+1}, x_n) \leq (a+b)\, d(Tx_n, Tx_{n+1}, x_n)$.

This yields (3) since $a+b < 1$.

Lemma 2. Let $T: X \to X$ satisfy (1) and (2) with $c > 0$. Then

$$d(x_{n+1}, Tx_{n+1}, u) \leq d(x_n, Tx_n, u) \;\forall\; n \geq 0 \text{ and } \forall\; u \in X,$$

where $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$.

Proof. From (4),

$$d(x_{n+1}, Tx_{n+1}, u) = d(Tx_n, Tx_{n+1}, u)$$

$$\leq a.\max\{d(x_n, Tx_n, u), d(x_{n+1}, Tx_{n+1}, u), (½)\, d(x_n, T_{x+1}, u)\}$$

$$+b.\max\{d(x_n, Tx_n, u), d(x_{n+1}, Tx_{n+1}, u)\} + c.d(x_n, Tx_{n+1}, u)$$

Using (iv) and (3)

$$d(x_{n+1}, Tx_{n+1}, u)$$

$$\leq (a+b).\max\{d(x_n, Tx_n, u), d(x_{n+1}, Tx_{n+1}, u)\} + c[d(x_n, Tx_n, u) + d(x_{n+1}, Tx_{n+1}, u)].$$

If $d(x_{n+1}, Tx_{n+1}, u) > d(x_n, Tx_n, u)$ for some n, them we have

$$d(x_{n+1}, Tx_{n+1}, u) < (a+b)\, d(x_{n+1}, Tx_{n+1}, u) + 2c\, d(x_{n+1}, Tx_{n+1}, u) = d(x_{n+1}, Tx_{n+1}, u),$$

a contradiction, and the proof is complete.

Remark. The relation

$$d(x_n, Tx_n, u) \leq d(x_0, Tx_0, u) \tag{5}$$

is immediate for any positive integer n.

Lemma 3. Let $T : X \to X$ be a mapping satisfying (1) where a,b,c satisfy (2) and $b > 0$, $c > 0$, then

$$d(x_n, Tx_n, u) \leq (1-bc)^{[n/2]} d(x_0, Tx_0, u),$$

where $[n/2]$ is the greatest integer not exceeding $n/2$.

Proof. Using (1) and Lemma 2, we have

$$d(x_{2k+1}, Tx_{2k+2}, u) = d(Tx_{2k}, Tx_{2k+2}, u)$$

$$\leq a. \max \{d(x_{2k}, Tx_{2k+1}, u), d(x_{2k}, Tx_{2k}, u),$$
$$(½)[d(x_{2k}, Tx_{2k+2}, u) + d(x_{2k+2}, Tx_{2k}, u)]\} + b.d(x_{2k}, Tx_{2k}, u)$$
$$+ c[d(x_{2k}, Tx_{2k+2}, u) + d(x_{2k}, Tx_{2k}, u)]. \tag{6}$$

By (3), (iv) and Lemma 2,

$$(½)[d(x_{2k}, Tx_{2k+2}, u) + d(x_{2k}, Tx_{2k}, u)]$$
$$\leq (½)[d(x_{2k}, Tx_{2k}, u) + d(x_{2k}, Tx_{2k+2}, Tx_{2k}) + d(Tx_{2k}, Tx_{2k+2}, u) + d(x_{2k}, Tx_{2k}, u)]$$
$$\leq 2d(x_{2k}, Tx_{2k}, u). \tag{7}$$

Using (7) in (6), we obtain

$$d(x_{2k+1}, Tx_{2k+2}, u)$$
$$\leq 2a\, d(x_{2k}, Tx_{2k}, u) + b.d(x_{2k}, Tx_{2k}, u) + 4c\, d(x_{2k}, Tx_{2k}, u)$$
$$\leq (2-b)\, d(x_{2k}, Tx_{2k}, u).$$

Using (1), Lemma 2 and (8), we have

$$d(x_{2k+2}, Tx_{2k+2}, u) = d(Tx_{2k+1}, Tx_{2k+2}, u)$$
$$\leq a. \max \{d(x_{2k}, Tx_{2k}, u), (½)d(x_{2k+1}, Tx_{2k+2}, u)\} + b.d(x_{2k}, Tx_{2k}, u)$$
$$+ c.d(x_{2k+1}, Tx_{2k+2}, u)$$
$$\leq a.d(x_{2k}, Tx_{2k}, u) + b.d(x_{2k}, Tx_{2k}, u) + c(2-b)\, d(x_{2k}, Tx_{2k}, u)$$
$$\leq (1-bc)\, d(x_{2k}, Tx_{2k}, u).$$

So

$$d(x_{2k+2}, Tx_{2k+2}, u) \leq (1-bc)^{k+1} d(x_0, Tx_0, u).$$

In view of Lemma 2 this gives

$$d(x_{2k+1}, Tx_{2k+1}, u) \leq d(x_{2k}, Tx_{2k}, u) \leq (1-bc)^k d(x_0, Tx_0, u).$$

Thus

$$d(x_n, Tx_n, u) \leq (1-bc)^{[n/2]} d(x_0, Tx_0, u),$$

and this is true for each $u \in X$.

The following concept on a-2metric space is due to Singh and Virendra [17]. Let T

be a mapping from a 2-metric space X into itself and x be a point in X. The mapping is said to be "asymptotically regular" at x if

$$\lim d(T^n x, T^{n+1}x, u) = 0 \text{ for all } u \in X.$$

Lemma 4. Let $T : X \to X$ be a mapping satisfying (1) wherein $a \geq 0,\ b > 0,\ c > 0$ and (2) holds. Then T is asymptotically regular at any point in X.

Proof. Let x_0 be an arbitrary point in X, then by Lemma 3, we have

$$d(T^n x_0, T^{n+1} x_0, u) = d(x_n, Tx_n, u) \to 0 \text{ as } n \to \infty.$$

Lemma 5. If T satisfies (1), where $a \geq 0,\ b > 0,\ c > 0$ and (2) holds, and if T has a fixed point p, then T is continuous at p.

Proof. Let $x_n \to p = Tp$. Then from (1),

$$\begin{aligned} d(Tx_n, Tp, u) &\leq a.\max\{d(x_n, Tp, u), d(x_n, Tx_n, u), (1/2)[d(x_n, Tp, u) + d(Tp, Tx_n, u)]\} \\ &\quad + b.d(x_n, Tx_n, u) + c[d(x_n, Tp, u) + d(p, Tx_n, u)] \\ &\leq (a+b)\,[d(x_n, Tp, u) + d(x_n, Tx_n, p) + d(Tp, Tx_n, u)] \\ &\quad + c\,[d(x_n, Tp, u) + d(Tp, Tx_n, u)]. \end{aligned}$$

Consequently

$$d(Tx_n, Tp, u) \leq [(1-2c)/c][d(x_n, Tp, u) + d(x_n, Tx_n, p) + d(x_n, Tp, u)].$$

From the above inequality it follows that $d(Tx_n, Tp, u) \to 0$ as $n \to \infty$.

So $Tx_n \to Tp$. and T is continuous at p.

Now we give our main result.

Theorem 1. Let (X,d) be a complete 2-metric space and $T : X \to X$ satisfy (1) and (2), where $a \geq 0,\ b > 0,\ c > 0$. Then T has a unique fixed point, and T is continuous at the fixed point.

Proof. Let $x = x_0 \in X$ be an arbitrary point. Define $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$.

By Lemma (2), we see that $\{T^n x\}$ is a Cauchy sequence. Since X is complete, there is a point $p \in X$ such that $\lim T^n x = p$.

Using (1), we have

$$d(T^n x, Tp, u)$$
$$\leq a. \max\{d(T^{n-1} x, p, u), d(T^{n-1} x, T^n x, u), d(p, Tp, u),$$
$$(1/2)[d(T^{n-1} x, Tp, u) + d(p, T^n x, u)]\}$$
$$+b. \max\{d(T^{n-1} x, T^n x, u), d(p, Tp, u)\} + c[d(T^{n-1} x, Tp, u) + d(p, T^n x, u)].$$

By taking the limit as $n \to \infty$,

$$d(p, Tp, u) \leq a\, d(p, Tp, u) + b.d(p, Tp, u) + c.d(p, Tp, u)$$
$$= (1-c)\, d(p, Tp, u).$$

This yields $d(p, Tp, u) = 0$ for each $u \in X$, and $Tp = p$.

Let p and q be two fixed points of T. Then by (1),

$$d(p, q, u) = d(Tp, Tq, u) \leq a\, d(p, q, u) + 2c\, d(p, q, u).$$

Therefore $0 \leq (1-a-2c)\, d(p, q, u) \leq 0$ for all $u \in X$, proving the uniqueness of the fixed point p. The continuity of T is immediate from Lemma 5.

Remarks.1. A slight generalization of Theorem 1 may be obtained by replacing T in (1) by T^m where m is a positive integer.

2. Notice that Theorem 1 is evidently true when (2) is replaced by $a + b + 2c < 1$. So our Theorem 1 extends and generalizes several known results. For example, an interesting result of Cho et al. [4, p. 103] is a particular case of Theorem 1. See also [12]and [13].

Now we give an application of the above Theorem.

Theorem 2. Let (X, d) be a complete 2-metric space, and T be a mapping from the product X × X into X such that

$$d(T(x, y), T(x^I, y^I), u)$$
$$\leq a. \max\{d(x, x^I, u), d(x, T(x, y), u), d(x^I, T(x^I, y^I), u),$$
$$(1/2)[d(x, T(x^I, y^I), u) + d(x^I, T(x, y,), u)]\}$$
$$+b. \max\{d(x, T(x, y), u), d(x^I, T(x^I, y^I,), u)\}$$
$$+c[d(x, T(x^I, y^I), u) + d(x^I, T(x, y), u)] \tag{9}$$

for all x, y, x', y', u in X, where $b > 0$ and a, c are non-negative real numbers such that $a + b + 2c = 1$. Then there exists exactly one point v in X such that $T(v, y) = v$ for all y in X.

Proof. By (9),

$$d(T(x, y), T(x', y), u)$$
$$\leq a.\ \max\{d(x, x', u), d(x, T(x, y), u), d(x', T(x', y), u),$$
$$(\tfrac{1}{2})[d(x, T(x', y), u) + d(x', T(x, y), u)]\}$$
$$+b.\ \max\{d(x, T(x, y), u), d(x', T(x', y), u)\}$$
$$+c[d(x, T(x', y), u) + d(x', T(x, y), u)]$$

for every x, x', y, u in X. Therefore by Theorem1, for each y in X, there exists one and only one $x(y)$ in X such that $T(x(y), y) = x(y)$.

For every y, y', by (9) we get

$$d(x(y), x(y'), u) = d(T(x(y), y), T(x(y'), y'), u)$$
$$\leq (a+2c)\ d(x(y), x(y'), u).$$

Consequently $x(y) = x(y')$, since u is arbitrary and $0 < a + 2c < 1$. Hence $x(.)$ is some constant $v \in X$, and so $T(v, y) = v$ for all y in X. Similar results under stronger conditions have been obtained in [11] and [15].

REMARKS ON HSIAO'S PAPER [2]

Chih-Ru Hsiao [2] made certain over enthusiastic observations regarding the study of contractive type mappings in 2-metric spaces. His conclusion has worked as a major hindrance in the development of fixed point theory in 2-metric/2-normed spaces. We show that his conclusion is worth overlooking.

1. Remarks of this paper apply to contraction or contraction type maps. This means contractive (in the sense of Edelstein) or contractive type maps and nonexpansive or nonexpansive type maps are not addressed by the author. We suspect that his remarks would apply to the later classes of maps.

2. Let (X, d) be a 2-metric space and $f, g : X \to X$. For any $x_0 \in X$, define $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$, $n = 0, 1, 2$ .... If $d(x_i, x_j, x_p) = 0$ for all nonnegative integers i, j, p then f and g are said to have property (H) in (X, d). (See definition 2[2]. This

defintion with $f = g$ is Definition 1 in [2].)

3. It is rightly observed in [2] that contraction or contraction type maps (called contractive type in [2]) satisfy the property (H), but the conclusion that it renders contraction type maps trivial is incorrect. This fact is evident from the following example.

4. **Example.** Let $X = [½, 1]^2$ and $d(x, y, z)$ be defined as the area of the triangle with vertices x, y, z in X. Let $f, g: X \to X$ be such that .

$f(a, b) = (a^{½}, b^{½})$ and

$$g(a,b) = \begin{cases} (a^2, b^2) & \text{if } a^2 \geq \frac{1}{2}, \quad b^2 \geq \frac{1}{2}, \\ (a^2, b) & \text{if } a^2 \geq \frac{1}{2}, \quad b^2 < \frac{1}{2} \\ (a,b) & \text{otherwise.} \end{cases}$$

Hasiao has shown in his Example (cf.[2], page 225) that f is nontrivial. It is shown in ([18], page 36) that g is nontrivial. Notice that, for $x_0 = (a, b)$, $½ \leq a, b \leq 1$, $x_1 = fx_0$, $x_2 = gx_1 = x_0$, etc. So $d(x_i, x_j, x_p) = 0$ for all nonnegative integers i, j, p and nontrivial maps f and g satisfy the property (H).

5. Indeed, for maps satisfying property (H), only the sequence of Picard iterates will be collinear in a 2-normed space but the maps need not be linear (or trivial). If we consider Jungck type iterative procedure, then (H) type property is satisfied not for the orbit but for the pre-orbit. So Hsiao's remarks (even if true) need not apply to Jungck type iterates.

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INTEGRAL REPRESENTATIONS OF THREE POLYNOMIALS LIKE HERMITE POLYNOMIALS

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ABSTRACT

We shall obtain the integral representations of three polynomials like Hermite polynomials. Also we obtain the generating functions of products of such Hermite polynomials.

AMS Subject Classifications (1990): 33E50.

Key words and Phrases : Generating functions, Hermite Polynomials, Integral representation.

INTRODUCTION

Rainville [2, p. 187] and Srivastava and Monocha [4,p. 33] defined the Hermite Polynomial $H_n(x)$ by the generating function.

$$exp.\ (2xt-t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{1.1}$$

valid for all x and t. Recently, modifying (1.1), Sinha [3, (104)] has defined three polynomials $H_{n,I}(x)$, $H_{n,II}(x)$ and $H_{n,III}(x)$ as

$$\exp\ (-2xt-t^2) = \sum_{n=0}^{\infty} \frac{H_{n,I}(x)t^n}{n!} \tag{1.2}$$

$$\exp\ (-2xt + t^2) = \sum_{n=0}^{\infty} \frac{H_{n,II}(x)t^n}{n!} \tag{1.3}$$

$$\exp\ (2xt-t^2) = \sum_{n=0}^{\infty} \frac{H_{n,III}(x)t^n}{n!} \tag{1.4}$$

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The Rodrigues' formula for the Hermite polynomials $H_n(x)$ [2, p. 189] is given by

$H_n(x) = (-1)^n \exp(x^2) D^n \exp(-x^2)$, $n=0, 1, 2, 3, \ldots\ldots\ldots\ldots$ (1.5)

Sinha [3, p. 106] has obtained the Rodrigue's type formula for polynomials $H_{n,I}(x)$, $H_{n,II}(x)$ *and* $H_{n,III}(x)$ respectively as

$H_{n,I}(x) = exp(x^2) D^n exp(-x^2)$. (1.6)

$H_{n,II}(x) = (-1)^n exp(-x^2) D^n exp(x^2)$ (1.7)

$H_{n,III}(x) = exp(-x^2) D^n exp(x^2)$. (1.8)

From (1.5) and (1.6)

$$H_{n,I}(x) = (-1)^n H_n(x)$$

From (1.7) and (1.8)

$$H_{n,II}(x) = (-1)^n H_{n,III}(x)$$

The explicit form of (1.1) and (1.4) are respectively,

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(2x)^{n-2k}}{k!\,(n-2k)!}$$

and

$$H_{n,III}(x) = \sum_{k=0}^{[n/2]} \frac{n!\,(2x)^{n-2k}}{k!\,(n-2k)!}$$

Thus, $H_n(x) \neq H_{n,III}(x)$ *for all* $n \geq 2$.

INTEGRAL REPRESENTATIONS

As in Lebedev *[1, p.63]*, we obtain the integral representations of $H_{n,I}(x)$, $H_{n,II}(x)$, $H_{n,III}(x)$ by using (1.6), (1.7) and (1.8) as

$$H_{n,I}(x) = \frac{2^n(-i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-t^2 - 2ixt\right) t^n dt, n = 0,1,2,3,\ldots\ldots\ldots \quad (2.1)$$

$$H_{n,II}(x) = \frac{2^n(-i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty} \exp\left(t^2 + 2ixt\right) t^n dt, n = 0,1,2,3,........ \tag{2.2}$$

$$H_{n,III}(x) = \frac{2^n(-i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(t^2 - 2ixt\right) t^n dt, n = 0,1,2,3,........ \tag{2.3}$$

Proof of (2.1): From Lebedev [1, p. 63] we have

$$e^{-x^2} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} \cos 2xtdt, \text{ for all values of } t \tag{2.4}$$

Differentiating w.r.t. x, using the Leibnitz's rule for differentiating under the integral sign, we get,

$$D^{2n}\left[\exp(-x^2)\right] = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (-1)^n 2^{2n} \exp\left(-t^2\right) t^{2n} \cos 2xt.dt$$

$$D^{2n+1}\left[\exp(-x^2)\right] = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (-1)^{n+1} 2^{2n+1} \exp\left(-t^2\right) t^{2n+1} \sin 2xt.dt$$

Using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and the fact that

$$\int_{-a}^{a} f_{even}(x)dx = 2\int_{0}^{a} f_{even}(x)\, dx.$$

and

$$\int_{-a}^{a} f_{odd}(x)dx = 0.$$

we arrive at

$$D^n\left[\exp(-x^2)\right] = \frac{(-i)^n 2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-t^2 - 2ixt\right) t^n . dt \qquad n = 0,\ 1,\ 2,\ \ldots\ldots \tag{2.5}$$

Multiplying by exp (x^2) on both the sides we get

$$e^{x^2} D^n\left(e^{-x^2}\right) = \frac{2^n (-i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^n e^{-t^{2-2xti}} dt,\ n = 0,\ 1,\ 2,\ \ldots\ldots \tag{2.6}$$

By (1.6) and (2.6), we obtain.

$$H_{n,I}(x) = \exp.\ (x^2) D^n\left(e^{-x^2}\right)$$

$$= \frac{2^n (-i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^n e^{-t^{2-2xti}} dt;\ n = 0,\ 1,\ 2,\ \ldots\ldots \tag{2.7}$$

Similarly we can obtain (2.2) and (2.3)

From the equations (2.1), (2.2) and (2.3) we can easily obtain generating function of product of the Hermite type polynomials.

$$\left(1-t^2\right)^{-1/2} \exp\left\{\frac{2xyt - \left(x^2+y^2\right)t^2}{\left(1-t^2\right)}\right\} = \sum_{n=0}^{\infty} \frac{H_{n,I}(x) H_{n,I}(y) t^n}{2^n n!}, |t| < 1 \tag{2.8}$$

$$\left(1-t^2\right)^{-1/2} \exp\left\{\frac{2xyt + \left(x^2+y^2\right)t^2}{\left(1-t^2\right)}\right\} = \sum_{n=0}^{\infty} \frac{H_{n,II}(x) H_{n,II}(y) t^n}{2^n n!}, |t| < 1 \tag{2.9}$$

$$\left(1-t^2\right)^{-1/2} \exp\left\{\frac{2xyt - \left(x^2+y^2\right)t^2}{\left(1-t^2\right)}\right\} = \sum_{n=0}^{\infty} \frac{H_{n,III}(x) H_{n,III}(y) t^n}{2^n n!}, |t| < 1 \tag{2.10}$$

Proof of (2.8) : By equation (2.1),

$$H_{n,I}(x)=\frac{2^n(-i)^n e^{x^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^n e^{-t^{2-2xti}} t^n dt;\ n=0,\ 1,\ 2,\ \ldots\ldots \tag{2.11}$$

On replacing x by y

$$H_{n,I}(y)=\frac{2^n(-i)^n e^{y^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^n e^{-t^{2-2yti}} .t^n dt;\ n=0,\ 1,\ 2,\ \ldots\ldots \tag{2.12}$$

Now, from equations (2.11) and (2.12), we have

$$\sum_{n=0}^{\infty}\frac{H_{n,I(x)}H_{n,I(y)}t^n}{2^n n!}=\sum_{n=0}^{\infty}\frac{2^n\left(i^2\right)^n e^{x^2+y^2}t^n}{\pi.n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-2iux-v^2-2ivy}(uv)^n du.dv.$$

Using the standard result, $\int_{-\infty}^{\infty}\exp\left(-v^2\right)=\sqrt{\pi}$, and the Maclaurin series of exp (θ), we arrive at

$$\sum_{n=0}^{\infty}\frac{H_{n,I}(x)H_{n,I}(y)t^n}{n!2^n}=\left(1-t^2\right)^{-1/2}\exp\left[\frac{2xyt-\left(x^2+y^2\right)}{\left(1-t^2\right)}\right]$$

Thus

$$\left(1-t^2\right)^{-1/2}\exp\left\{\frac{2xyt-\left(x^2+y^2\right)t^2}{\left(1-t^2\right)}\right\}=\sum_{n=0}^{\infty}\frac{H_{n,I}(x)H_{n,I}(y)t^n}{2^n n!},|t|<1 \tag{2.13}$$

Similarly we can obtain (2.9) and (2.10)

Put $y = x$ in (2.8), (2.9) and (2.10) we obtain the following fruitful generating functions:

$$\left(1-t^2\right)^{-1/2} exp\left\{\frac{2x^2t}{(1+t)}\right\}=\sum_{n=0}^{\infty}\frac{H^2{}_{n,I}(x)t^n}{2^n n!},|t|<1 \tag{2.14}$$

$$\left(1-t^2\right)^{-1/2}\exp\left\{\frac{2x^2t}{(1-t)}\right\}=\sum_{n=0}^{\infty}\frac{H^2{}_{n,II}(x)t^n}{2^n n!},|t|<1 \tag{2.15}$$

$$\left(1-t^2\right)^{-1/2}\exp\left\{\frac{2x^2t}{(1-t)}\right\}=\sum_{n=0}^{\infty}\frac{H^2{}_{n,III}(x)t^n}{2^n n!},|t|<1 \tag{2.16}$$

CONCLUSION

By applying Liebnitz's theorem and method given in Lebedev [1] for integral representation of Hermite polynomials. We obtain integral representations and generating functions of three polynomials like Hermite polynomials.

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SEASONAL ABUNDANCE OF PHYTOPLANKTON IN THE LOTIC ECOSYSTEM OF ALAKNANDA, GARHWAL HIMALAYA

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ABSTRACT

Phytoplankton distribution and abundance in the Alaknanda River within the 22 km. stretch of the Garhwal Himalayas were investigated from January to December, 1997. A total of 28 genera and 89 species of phytoplankton belonging to the families-Bacillariophyceae, Chlorophyceae, Desmidiaceae and Myxophyceae were identified. These comprised diatoms (91.72%), green algae (4.09%), desmids (0.86%) and blue green algae (1.22%). The monthly population fluctuations indicated that the Alaknanda river had the highest phytoplankton in March (1620 units/1) with a decreasing trend onwards and reached lowest in August (60 units/1). Phytoplankton abundance and species richness appeared to be influenced by high turbidity, total alkalinity, fluctuating water levels and dissolved oxygen.

Key words: Phytoplankton; River Alaknanda; Algae

INTRODUCTION

The phytoplankton of high altitude cold water are most distinct than those of any other type of aquatic habitat and include a large percentage of species which are restricted to this particular habitat. These provide the main food item of fishes directy or indirectly and can be used as indicator of the trophic phase of the water body [25]. Phytoplankton abundance is controlled by several physico-chemical factors of water. A number of researches suggests that light is often a limiting factor for phytoplankton growth [14, 16, 17, 24]. According to Crayton and Sommerfeld [7] phytoplankton abundance and species richness apeared to be influenced by high turbidity, current velocity, fluctuating water level and age of the water. Alkalinity also plays an important role to monitor the potential of the phytoplankton [9, 23]. Phytoplankton requires some essential inorganic nutrients to grow well in addition to carbon, hydrogen and oxygen [21,28]

Badola and Singh [3], Baduni [4], Nautiyal [17], Negi [16] and Nautiyal et. al. [18] have contributed on some aspects of hydrobiology and primary productivity of this river. The present investigation was aimed at determining quantitative composition of phytoplankton in Alaknanda.

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STUDY AREA

The river Alaknanda, a parent stream of the holy river Ganges, originates from the 'Satopanth Bhagat Kharak' group of glaciers confined to the eastern slope of the Chaukhamba mountain of Hamalayan ranges. The river Alaknanda confluence with Bhagirathi river at Deoprayag (442 m above m.s.l.) after passing 191 km of its course. After the confluence of Alaknanda and Bhagirathi, the river is called as Ganges. The Alaknanda in its whole longitudinal sections shows an average of 17.3 m/km gradient. The gradient of the study area was recorded to be 3.3 m/km which provides relatively more suitable environment for phytoplankton.

Three sampling sites have been selected at the Alaknanda river. The first (Kaliyasaur) is located on the latitude 30^0 15'N and longitude 78^0 53',8"E at a height of 580 m from sea level on the left bank of the river Alaknanda. This spot has terrace like topography and the slope was thus very gentle. The mature boulders of various sizes constitute the river bed and banks. The second S_2 (Chauras) is located on the latitude 30^0 13' 10"N and longitude 78^0 48' 10"E at a height of 550 m from sea level on the left bank of the river at the outskirts of the town just below a pedestrian bridge on river Alaknanda, immediately upstream of which is located Srikot. Before this site, the river meanders twice first to left and then to right at almost 90^0. A chiff like topography occurs on both the side of the river thus human interference is least, although just upstream and immediately downstream, the human activity (daily and recreational) tends to increase. Rocks were the major substratum. Prismatic boulders of small to medium size were scantly scattered. The third S_3 (Bilkedar) is located on the latitude 30^0 10' 13"N and longitude 78^0 44' 55"E at a height of 538 m from sea level, 4 km towards west of Srinagar township. The sampling spot is selected on the left bank at the confluence of Bilkeder Gad and river Alaknanda. The channel bed is filled with boulders, pebbles, weathered material and rock substratum. This spot is situated adjacent to the village so that a lot of daily and recreational activity occurs all around.

MATERIALS AND METHODS

Monthly sampling was conducted from three sampling stations for recording physico-chemical parameters and estimating phytoplankton density. The water (100 liters) was sieved through a number 20 plankton net, concentrated into a 60 ml vial and preserved in 5% formaldehyde or 70% ethanol. 60 ml samples were concentrated to 20 ml by centrifugation. A Hensen-Stempel pipet was used to take 1 ml. aliquots into four Sedgewick Rafter counting chambers. Each cell was then examined under microscope for identification and counting. The phytoplanktonic identification was done following Welch [27] and Ward and Whipple [26].

The temperature was recorded with the help of a graduated mercury thermometer and pH was determined by portable pH meter (control dynamic pH meter model APx 175 E/C). Turbidity was recorded with the help of a Nephalometer (Elico model 1-52). The velocity of water current was measuered with the help of a current meter and Robin Growford method. The physico-chemical parameters were monitored following APHA [1].

RESULTS AND DISCUSSION

A total of 28 genera and 89 species of phytoplankton belonging to the families Bacillariophyceae, Chlorophyceae, Desmidiaceae and Myxophyceae were recorded during the period of investigation (Table-1). The largest and most diverse group was the diatoms (Bacillariophyceae), which contributed 91.72% of the total phytoplankton. *Achnanthes, Cymbella, Diatoma, Navicula, Nitzschia* and *Synedra* were the dominant genera among diatoms. Green algae (Chlorophyceae) 4.09% blue-green algae (Myxophyceae) 1.22% and desmids (Desmidiaceae) 0.86% ranked second, third and fourth respectively in order of their abundance. However, their number was far less when compared with the diatoms Miscellaneous algae also contributed 2.20% of the total phytoplankton population (Table-2)

The important green algae genera recorded were *Hydrodictyon, Cladophora, Microspora, Spirogyra, Tetraspora* and *Ulothrix.* Desmids were represented by three genera *Closterium, Cosmarium* and *Gonatozygon* and blue-green algae by *Anabaena, Nostoc, Oscillatoria, Polycystis* and *Rivularia. Among the chlorophyceae,* cladophora glomerata was found to be absent during winter due to low temperature. *Spirogyra* and *Ulothrix variabilis* and *U.zonata* were found to be absent during monsoon due to increased turbulence which consequently leads to detachment of algal filaments from the substratum. Similar observations were recorded by Sharma [22] on river Bhagirathi, Nautiyal [17] on river Ganga and Gusain [10] on the river Bhilangan of Garhwal Himalaya.

In the River Alaknanda winter samples revealed mean maximum phytoplankton population abundance of 247.54±473.68 units/l when turbidity (6.22±6.19 NTU) and water velocity (0.310±0.009m/s) were low. The minimum mean value of phytoplankton (17.34±36.93 units/1) occur during monsoon months (July-August), which may be due to high percentage of periodic turbidity (420.45±100.80 NTU) and high current velocity (3.718±0.08 m/s). Sehgal [20] observed plankton density from 13 to 11643 units/l in 1985 and 3 to 20896 units/l in 1986 in River Beas of Himachal Pradesh. Baruah *et.al.* (2) recorded maximum plankton population (4548 units/1) in winter and minimum (1425 units/1) in monsoon in river Brahmaputa of Assam.

According to Hynes, [11] water movement, turbidity, temperature and nutrients are

the main environmental factors which control the abundance of plankton. Swale [19] observed in the River Lee (U.K.) that the low water temperature and high light intensity are suitable to the planktonic growth in the river. In the present study on Alanknanda, the proliferation of phytoplankton from winter to summer could be attributed to the progressively increasing water temperature and photoperiod (14 hrs.)

Turbidity affects the number of phytoplankton. Ellis [8] states that erosive silt in rivers acts as an opaque screen to all wavelengths of light not allowing the phytoplankton to carry out photosynthesis. Chandler [6] and Cushing [5] report that mechanical destruction of plankton occurs by the grinding action of water heavily laden with silt. The lowest phytoplankton numbers (60-120 units/1) in the Alaknanda river during monsoon months (July-August) may be due to high percentage of periodic turbidity and large suspended sediment load of high current velocity as a result of heavy precipitation in the upper catchment area. Keithan and Lowe [12] reported low densities in fast flow area and high densities in slow flow area. Lamb [15] also reported high cell densities is slow current communities. It is concluded from the above discussion that turbidity is a lethal factor and water flow is a detrimental factor which together exhibit plankton development.

Table 1. List of phytoplankton occuring in the river Alaknanda

Bacillariophyceae
Achanthes affinis
A. bisoletiana
A. brevipes
A. Brevipes var intermedia
A. fragilaroides
A. lacunarum
A. lanceolata
A. lanceolata f. capitata
A. lanceolata var rostrata
A. linearis
A. microcephala
A. minutissima var cryptocephala
Amphora ovalis
A. ovalis var pediculus
A. veneta
Coconies placentula
C.placentula var euglypta
C.pediculus
Cymbella affinis
C.brehmii

C.gonzalvesii
C.hustedtii
C.laevis
C.nagpurensis
C.parva
C.tumida
C.tumidula
C.turgida
C.turgidula
C.ventricosa
Diatoma anceps
D.heimale
D.vulgare
Frazilaria capucina
F. capitata
F.construens
F.intremedia
F.lapponica
F.pinnata
G.angustatum
G.angustatum var producta
G.longiceps
G. longiceps var subclavata
G. lanceolatum
G. nagpurensis
G. olivaceum
G. olivaceum var calcarea
G. parvulum
Gyarosigma spp.
Hantzschia amphioxys
H. amphioxys f. capitata
Navicula bacillum
N. grimmi
N. radiosa
N.radiosa var tenella
N.rhyncocephala
N.rhyncocephala var grunowii
N. rostellata
N. salinarum
N. viridula
Nitzschia amphibia

N. capitellata
N. fonticola
N. linearis
N. microcephala
Pinnularia spp
Rhoicosphenia curvata
R.vermicularis
Synedra acus
S. acus var angustissima
S.rumpens
S.ulna
S.ulna var oxyrhynchus
S. ulna var oxyrhynchus forma mediocontracta

Chlorophyceae
Cladophora glomerata
Hydrodictyon
Microspora
Spirogyra
Tetraspora
Ulothrix variabilis
U. zonata

Desmidiaceae
Cosmarium monomazum
Closterium teibleiinii
Gonatozygon

Myxophyceae
Anabaena
Nostoc
Oscillatoria
Polycystis
Rivularia

Table 2. Monthly mean flucturations of phytopiankton (units/I) in the Alaknanda river

Months	Bacillariophyceae	Chlorophyceae	Desmidiaceae	Myxophyceae	Miscellaneous	$\bar{x} \pm$ SD
Jan	800 ± 50.00	33.34 ± 15.28	16.67 ± 11.55	6.67 ± 11.55	30.00 ± 10.00	177.34 ± 348.25
Feb	766.67 ± 124.24	50.00 ± 10.00	10.00 ± 10.00	16.67 ± 5.78	10.00 ± 10.00	170.67 ± 333.59
Mar	1094.34 ± 274.66	60.00 ± 10.00	13.34 ± 5.78	30.00 ± 20.00	40.00 ± 36.06	247.54 ± 473.68
Apr	596.67 ± 60.28	26.67 ± 5.78	3.34 ± 5.78	NIL	10.00 ± 17.33	127.34 ± 262.57
May	343.34 ± 198.58	13.34 ± 5.78	NIL	NIL	NIL	71.34 ± 152.17
Jun	216.67 ± 89.63	10.00 ± 10.00	NIL	NIL	NIL	45.34 ± 95.88
Jul	150.00 ± 70.00	NIL	NIL	NIL	NIL	30.00 ± 67.09
Aug	83.34 ± 25.17	NIL	NIL	NIL	3.34 ± 5.78	17.34 ± 36.93
Sep	133.34 ± 32.15	6.67 ± 5.78	NIL	NIL	10.00 ± 10.00	30.00 ± 57.93
Oct	363.34 ± 169.22	10.00 ± 10.00	NIL	NIL	33.34 ± 41.64	81.34 ± 158.24
Nov	463.34 ± 155.03	26.67 ± 5.78	3.34 ± 5.78	20.00 ± 10.00	6.67 ± 5.78	104.00 ± 201.10
Dec	706.67 ± 64.29	16.67 ± 5.78	6.67 ± 5.78	6.67 ± 5.78	6.67 ± 5.78	148.68 ± 311.97
%	91.12	4.09	0.86	1.22	2.20	

Table 3. Monthly mean variations in the Physico-chemical parameters of all the three sampling sites in the river Alaknanda, Garhwal Himalaya

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	$\bar{x}$ ± S.D.
Air Temperature (°C)	12.50 ± 1.50	16.50 ± 0.50	21.00 ± 1.00	22.40 ± 0.52	24.00 ± 1.50	25.67 ± 1.89	25.17 ± 1.89	23.84 ± 2.25	22.83 ± 1.44	20.06 ± 0.90	18.00 ± 1.73	15.00 ± 2.64	20.59 ± 4.25
Water Temperature (°C)	11.00 ± 1.32	13.16 ± 0.76	15.50 ± 0.87	17.34 ± 1.16	18.84 ± 0.58	18.34 ± 0.58	17.67 ± 0.77	16.84 ± 0.77	15.66 ± 0.29	13.34 ± 2.09	11.84 ± 1.26	9.70 ± 0.58	14.94 ± 3.06
Turbidity (NTU)	6.22 ± 6.19	7.44 ± 6.62	10.11 ± 4.85	14.11 ± 4.72	41.45 ± 29.06	63.09 ± 45.03	420.45 ± 100.80	57.41 ± 27.91	30.20 ± 10.67	16.21 ± 15.43	13.65 ± 13.00	9.98 ± 10.54	137.75 ± 402.99
Current Velocity (m/sec)	0.405 ± 0.014	0.310 ± 0.009	0.325 ± 0.005	0.677 ± 0.007	1.063 ± 0.034	1.820 ± 0.010	3.213 ± 0.010	3.718 ± 0.008	2.048 ± 0.010	1.1231 ± 0.008	0.890 ± 0.005	0.810 ± 0.014	1.376 ± 1.123
Conductivity (µ S)	160.60 ± 1.01	180.34 ± 10.42	175.34 ± 11.25	153.54 ± 11.75	150.64 ± 12.12	146.06 ± 7.73	139.16 ± 9.54	144.03 ± 14.11	145.46 ± 14.75	152.20 ± 22.41	169.97 ± 13.36	165.60 ± 14.10	156.66 ± 13.57
pH	8.16 ± 0.20	8.64 ± 0.20	8.07 ± 0.26	7.86 ± 0.36	8.03 ± 0.07	8.32 ± 0.27	8.19 ± 0.20	7.62 ± 0.45	7.87 ± 0.04	7.93 ± 0.20	8.04 ± 0.01	8.23 ± 0.34	8.07 ± 0.23
Dissolved O_2 ($g.\ l^{-1}$)	10.07 ± 0.37	9.60 ± 0.36	9.20 ± 0.43	8.50 ± 0.86	8.57 ± 0.20	8.14 ± 0.15	7.17 ± 0.55	8.04 ± 0.96	8.00 ± 1.05	7.90 ± 0.10	8.70 ± 0.55	9.37 ± 0.45	8.60 ± 0.83
Free CO_2 (mgl^{-1})	0.34 ± 0.12	0.44 ± 0.09	0.46 ± 0.10	0.54 ± 0.06	0.52 ± 0.04	0.55 ± 0.04	0.69 ± 0.02	0.55 ± 0.13	0.58 ± 0.07	0.60 ± 0.04	0.51 ± 0.07	0.42 ± 0.12	0.52 ± 0.09
Total alkalinity ($mg.l^{-1}$)	82.19 ± 4.54	76.84 ± 4.26	71.34 ± 4.17	63.19 ± 2.29	54.39 ± 5.11	49.34 ± 5.86	43.89 ± 5.35	37.59 ± 2.77	42.85 ± 4.52	51.84 ± 6.25	58.67 ± 11.59	70.09 ± 7.09	58.52 ± 14.34
Nitrates ($mg.l^{-1}$)	0.0020 ± 0.001	0.0025 ± 0.002	0.0027 ± 0.003	0.0034 ± 0.003	0.0054 ± 0.004	0.0061 ± 0.005	0.0109 ± 0.002	0.0099 ± 0.003	0.0075 ± 0.003	0.0058 ± 0.003	0.0046 ± 0.002	0.0029 ± 0.002	0.0053 ± 0.002
Phosphates ($mg.l^{-1}$)	0.0057 ± 0.044	0.0117 ± 0.003	0.0144 ± 0.004	0.0180 ± 0.005	0.0260 ± 0.005	0.0307 ± 0.008	0.0300 ± 0.013	0.417 ± 0.008	0.0450 ± 0.015	0.0324 ± 0.009	0.024 ± 0.005	0.0160 ± 0.005	0.024 ± 0.02

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A STUDY OF THE LINE EXPLOSION IN RADIATIVE MAGNETOGASDYNAMICS

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ABSTRACT

Similarity solutions describing the flow of a perfect gas behind cylindrical shock waves with radiation heat flux and transverse magnetic field in a rotating non-uniform atmosphere are investigated. The total energy of the expanding wave has been assumed to remain constant. The density in the undisturbed conducting gas medium is suppose to obey a power law. The solution, however, is only applicable to a gaseous medium where the undisturbed pressure falls as the inverse square of the distance from the line of explosion occurring in the rotating atmosphere.

Key words : Non-homogeneous, rotating atmosphere shock wave, radiation flux,transverse magnetic field.

Classification Numbers : PACS 52.53 and 52.35 Bj.

INTRODUCTION

A cylindrical wave of line explosion with a shock surface as wave front, produced on account of a sudden release of a finite amount of energy, expanding outwards in a conducting gas subjected to a magnetic field, has been studied by Pai [1], Greenspan [2], Christer and Helliwell [3],Singh and Vishwakarma [4], and many other.

Elliot[5] studied self-similar solutions for spherical blast waves in air using Rosseland's diffusion approximation under the assumption of nonexistence of heat flux at the centre of symmetry. Wang[6]and Helliwell [7] have considered the shock wave problems with thermal radiation using similarity method of Sedov [8] in ordinary gasdynamics. Singh [9,10] has discussed the cylindrical shock wave with radiation heat flux using similarity method. Singh and Vishwakarma [11,12] have investigated the self-similar flows behind cylindrical shock waves in radiative magnetogasdynamics. Gretler and Steiner [13] and Gretler [14] have also discussed blast waves in inhomogenous media taking into consideration the effects of counter pressure and heat transfer in detail without considering the effects of magnetic field. Recently Singh et al.[15] have studied a model of magnetoradiative strong shock wave in uniform atmosphere using similarity method.

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In the present paper we have considered the cylindrical wave of explosion with a shock surface as wave front, produced on account of a sudden release of a finite amount of energy, expanding outwards in a conducting rotating atmosphere with radiantion flux under the assumption of isothermal shock conditions. For the existence of self-similar character, we have assumed that both radiation pressure and energy are negligible. We have taken the shock transparent and isothermal, and the gas to be grey and opaque. The total energy of explosion is constant. The results of numerical calculations for different models are illustrated through figures. Moreover, a comparative study has been made between our results and the results obtained by Singh [9,10].

EQUATIONS OF MOTION AND SHOCK CONDITIONS

The cylindrical polar coordinates, where r is the radial distance from the line of explosion, have been used here. The equation of conservation of mass, momentum, transverse velocity, transverse magnetic field and energy behind the wave in rotating gas are

$$\frac{\partial \rho}{\partial t}+\upsilon\frac{\partial \rho}{\partial r}+\frac{\rho}{r}\frac{\partial}{\partial r}(ur)=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{r}\left[\frac{\partial p}{\partial r}+\frac{m}{2}\frac{\partial h^2}{\partial r}+\frac{mh^2}{r}\right]-\frac{v^2}{r}=0 \tag{2}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)(vr)=0 \tag{3}$$

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial r}(hu)=0 \tag{4}$$

$$\frac{\partial E}{\partial t}+u\frac{\partial E}{\partial r}+p\left[\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)+u\frac{\partial}{\partial r}\left(\frac{1}{p}\right)\right]+\frac{1}{\rho r}\frac{\partial}{\partial r}(rq)=0 \tag{5}$$

where ρ is the density; u, the radial velocity; p, the pressure; h, the transverse magnetic field; v, the transverse velocity; E, the internal energy; μ, the magnetic permeability; q, the heat flux; and t, the time.

For an ideal gas

$$E=\frac{p}{(\gamma-1)\rho}, p=\Gamma\rho T, \tag{6}$$

where γ is the adiabatic gas index; T, the temperature; and Γ, the gas constant.

Assuming local thermodynamic equilibrium and taking Rosseland's diffusion approximation we have

$$q=\frac{-c\nu}{3}\frac{\partial}{\partial r}(\sigma T^4), \tag{7}$$

where $\frac{1}{4}\sigma c$ is the Stefan-Boltzmann constant; c, the velocity of light; and ν, the mean free path of radiation which is a function of density and temperature, given as

$$\nu=\nu_0\rho^\alpha T^\beta, \tag{8}$$

where ν_0, α and β are constants. The disturbance is headed by an isothermal shock and the conditions are

$$\rho_2(V\text{-}u_2)=\rho_1 V=m_s, \tag{9}$$

$$\left(p_2+\frac{\mu h_2^2}{2}\right)-\left(p_1\frac{\mu h_1^2}{2}\right)=m_s u_2, \tag{10}$$

$$E_2+\frac{p_2}{\rho_2}+\frac{\mu h_2^2}{\rho}+\frac{1}{2}(V-u_2)^2-\frac{q^2}{m_s}=E_1+\frac{p_1}{\rho_1}+\frac{\mu h_1^2}{\rho_1}+\frac{1}{2}V^2, \tag{11}$$

$$T_2=T_1, \tag{12}$$

$$\nu_2=\nu_1, \tag{13}$$

where subscripts 2 and 1 are for the regions just behind and just ahead of the shock surface, respectively, and V denotes the shock velocity.

In front of the shock in the undistrubed gaseous medium, the density, pressure and magnetic field distributions are as given below

$$\rho_1 = \rho_c R^{\omega}, \quad -2 < \omega < 0, \tag{14}$$

$$p_1 = p_c R^{n}, \quad n < 0, \tag{15}$$

$$h_1 = h_c R^{\delta}, \quad \delta < 0, \tag{16}$$

where R is the shock radius at the time t. ρ_c, p_c, h_c, ω, n, and δ are constants.

We know

$$\frac{v}{r} = \omega_0, \tag{17}$$

where ω_0 is angular velocity.

Let us suppose

$$\omega_0 = Bt^{1}, \tag{18}$$

according to dimensional consideration, B being a constant.

SOLUTIONS OF THE EQUATIONS

Let us seek solutions of equations (1) to (5) in the form

$$\left.\begin{aligned}
u &= \frac{r}{t}U(\eta), \\
v &= \frac{r}{t}S(\eta), \\
\rho &= r^{k}t^{\lambda}\Omega(\eta), \\
p &= r^{k+2}t^{\lambda-2}P(\eta), \\
\sqrt{\mu}h &= r^{(k+2)/2}t^{(\lambda-2)/2}H(\eta), \\
q &= r^{k+3}t^{\lambda-3}F(\eta),
\end{aligned}\right\} \tag{19}$$

where

$$\eta = r^{a}\, t^{b}, \tag{20}$$

while k, η, a and b are constants and $U(\eta)$ is non-dimensional velocity.

The total energy of the disturbance per unit length is

$$Q = 2\pi \int_0^R \left(\frac{1}{2}\rho u^2 + \frac{P}{(\gamma - 1)} + \frac{\mu h^2}{2} \right) r dr. \tag{21}$$

In terms of variable η, we get

$$Q = 2\pi \int_\infty^{\eta_0} \left(\frac{1}{2}\Omega U^2 + \frac{P}{(\gamma - 1)} + \frac{1}{2} H^2 \right) \eta^{[\{(k+4)/a\}-1]} t^{[\lambda-2-(a/b)(k+4)]} d\eta, \tag{22}$$

where η_0 is the value of η at shock front.

We choose the shock surface to be given by η_0 = constant. This choice fixes the velocity of the shock surface as,

$$V = -\frac{b}{a}\frac{R}{t}, \tag{23}$$

which represents an outgoing surface if $a < 0$. The total energy of disturbance within the shock surface at any time t is constant. This by equation (22) requires that

$$\lambda - 2 - \frac{b}{a}(k + 4) = 0 \tag{24}$$

Let the Mach number and Alfve'n Mach number at the shock front be defined respectively, as

$$M^2 = \frac{V^2 \rho_1}{\gamma p_1}, \tag{25}$$

and

$$M^2{}_A = \frac{V^2 \rho_1}{\mu h_1^2} \tag{26}$$

By direct substitution of equations (19) in equations of motion (1) to (8), shock conditions (9) to (13) and after utilising the relation (24) we find that similarity conditions, following Singh [9], are compatible when

$$\left.\begin{array}{l} k = \omega, \quad \lambda = 0, \quad a = -(4+\omega), \quad b = 2 \\ n = -2 \quad \beta = -\dfrac{5}{2}, \quad \alpha = \dfrac{\omega+1}{\omega} and \quad \delta = -1 \end{array}\right\} \tag{27}$$

Hence, the pressure and magnetic field distributions become

$$p_1 = p_c R^{-2} \tag{28}$$

and

$$h_1 = h_c R^{-1}$$

Now after substituting equations (19 and (20) into equations of motion (1) to (5) and (7), using (27), we get

$$\frac{\Omega' \rho_c}{\Omega / \rho_c} = \frac{\eta(4+\omega)U' - (2+\omega)U}{\eta[2-(4+\omega)U]}, \tag{30}$$

$$\frac{P'/\rho_c}{P/\rho_c} = \frac{\Omega/\rho_c}{P/\rho_c}\left[\frac{\eta U'\{2-(4+\omega)U\}+\{U(U-1)-S^2\}}{\eta(4+\omega)}\right] + \frac{H^2/\rho_2}{P/\rho_c}\left[\frac{1}{\eta} - \frac{2H'}{H}\right] + \frac{\omega+2}{\eta(4+\omega)}, \tag{31}$$

$$\frac{S'}{S} = \frac{(1-2U)}{\eta[2-(4+\omega)U]}, \tag{32}$$

$$\frac{H'/\rho_c}{H/\rho_c} + \frac{\left[\eta(4+\omega)U' - \dfrac{4+\omega}{2}\right]U + 1}{\eta[2-(4+\omega)U]}, \tag{33}$$

$$\frac{F'/\rho_c}{F/\rho_c} = \frac{P'/\rho_c}{F/\rho_c}\left[\frac{\{2-(4+\omega)U\}}{(\gamma-1)(4+\omega)}\right] + \frac{P/\rho_c}{F/\rho_c}\left[\frac{(\omega+2\gamma+2)U - \gamma\eta(4+\omega)U' - 2}{\eta(\gamma-1)(4+\omega)}\right] + \frac{1}{\eta}, \tag{34}$$

$$\frac{F/\rho_2}{P/\rho_2} = -N\frac{\left(\dfrac{P}{\rho_c}\right)^{\frac{1}{2}}}{\left(\dfrac{\Omega}{\rho_c}\right)^{\left(\frac{3}{2}-\alpha\right)}}\left[2 - \eta(4+\omega)\left\{\frac{\dfrac{P'}{\rho_c}}{\dfrac{P}{\rho_c}} - \frac{\dfrac{\Omega'}{\rho_c}}{\dfrac{\Omega}{\rho_c}}\right\}\right] \tag{35}$$

where

$$N = \frac{4\sigma c \nu_0}{3\Gamma^2 \rho_c^{(1-\alpha)}} = \text{ a non- dimensional parameter,}$$

and

$$U' = \frac{\frac{1}{N}\frac{F/\rho_c(\Omega/\rho_c)^{\left(\frac{1}{2}-\alpha\right)}}{(P/\rho_c)^{\frac{1}{2}}}\left[2-(4+\omega)U\right] - 2\frac{P/\rho_c}{\Omega/\rho_c}\left[\omega+(4+\omega)U\right] - \left[2-(4+\omega)U\right]\left[U(U-1)-S^2\right]}{\eta\left[\left\{2-(4+\omega)U\right\}^2 \frac{(P+2H^2)/\rho_c}{\Omega/\rho_c}\right]}$$

(36)

where prime denotes the differentiation with respect to η.

The approximate shock conditions after transformation yield.

$$U(\eta_o) = \frac{2\left(1-\frac{1}{x}\right)}{(4+\omega)}, \tag{37}$$

$$\frac{\Omega(\eta_0)}{\rho_c} = x, \tag{38}$$

$$\frac{p(\eta_o)}{\rho_c} = \frac{4x}{\gamma(4+\omega)^2 M^2}, \tag{39}$$

$$S(\eta_0) = B, \tag{40}$$

$$\frac{H(\eta_0)}{\rho_C} = \frac{2x}{(4+\omega)M_A}, \tag{41}$$

$$\frac{F(\eta_0)}{\rho_c} = \frac{8(x-1)}{(4+\omega)^3}\left[\frac{1}{M^2{}_A} - \frac{(x-1)}{2x^2}\right], \tag{42}$$

where x is obtained from the following relation

$$x = -\left(A + \frac{1}{2}\right) + \left[\left(A + \frac{1}{2}\right)^2 + 2\gamma M_A^{\ 2}\right]^2, \tag{43}$$

while

$$A = \frac{M_A^{\ 2}}{\gamma M^2}, \tag{44}$$

which are the initial values for our numerical calculation, where we assume that $\eta_0=1$.

RESULTS AND DISCUSSION

Differential equations (30) to (36) are numerically solved by the Runge-Kutta technique and the solutions are represented in a convenient form as

$$\frac{u}{u_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{1}{(4+\omega)}} \frac{U(\eta)}{U(\eta_0)}, \tag{45}$$

$$\frac{\rho}{\rho_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{\omega}{(4+\omega)}} \frac{\Omega(\eta)}{\Omega(\eta_0)}, \tag{46}$$

$$\frac{P}{P_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{(2+\omega)}{(4+\omega)}} \frac{P(\eta)}{P(\eta_0)}, \tag{47}$$

$$\frac{v}{v_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{1}{(4+\omega)}} \frac{S(\eta)}{S(\eta_0)}, \tag{48}$$

$$\frac{h}{h_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{(2+\omega)}{2(4+\omega)}} \frac{H(\eta)}{H(\eta_0)}, \tag{49}$$

and

$$\frac{q}{q_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{(2+\omega)}{(4+\omega)}} \frac{F(\eta)}{F(\eta_0)}, \tag{50}$$

The numerical results for a certain choice of parameters are reproduced in the form of figures. Nature of the field variable may be illustarted through figures 1 to 6. We calculate our results for the following two sets of parameters:

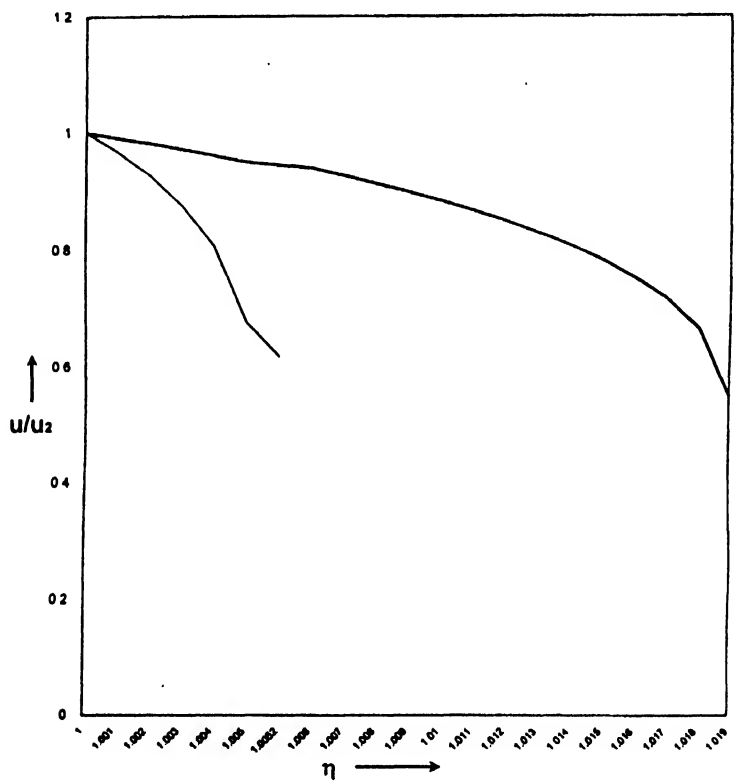

Fig. 1. Distribution of radial velocity

(i) $\gamma = \frac{4}{3}, \quad N = 10, \quad \omega = -1.5, \quad \alpha = \frac{1}{3}, \quad B = 5$

(ii) $\gamma = \frac{5}{3}, \quad N = 100, \quad \omega = -1.75, \quad \alpha = \frac{3}{7}, \quad B = 10$

For both the cases we take M_2=20 and $M^A{}_2$=10. Similarily we can take other different models for calculation.

In the figures the results of first set (i) are shown through dark and thick line and second set (ii) with light and thin line.

The range of flow variables is small. Radial velocity, density, pressure and magnetic field have decreasing trend throughout the flow but heat flux is of increasing order. Transverse velocity has constant magnitude. Heat radiation is very high due to rotation for both sets.

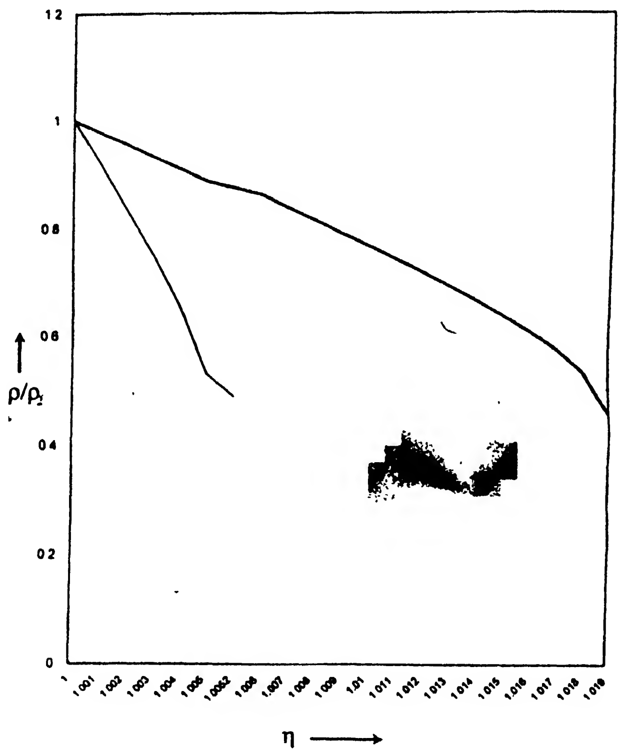

Fig. 2. Distribution of density

If we compare our results with the paper of Singh [10] and Singh [9], we find that our range of flow variables are very small. In ordinary gas dynamics without rotation radiation flux in first set increases but in second set decreases while in magnetogasdynamics without rotation radiation flux increase for first set and for second set it first decreases and then increases. But in the present problem, it increases sharply for both the sets.

The distribution of magnetic field in the case of magnetogasdynamics in the absence of rotating gas is of increasing order, but in the present problem it always decreases for every sets.

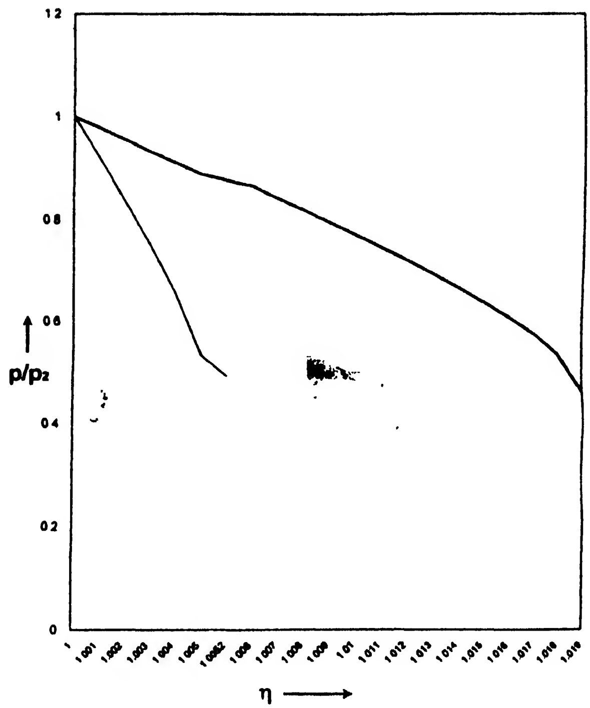

Fig. 3. Distribution of pressure

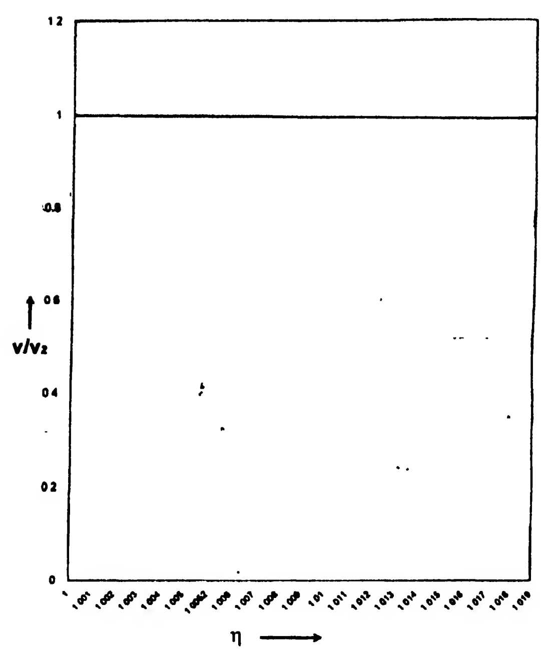

Fig. 4. Distribution of transverse velocity

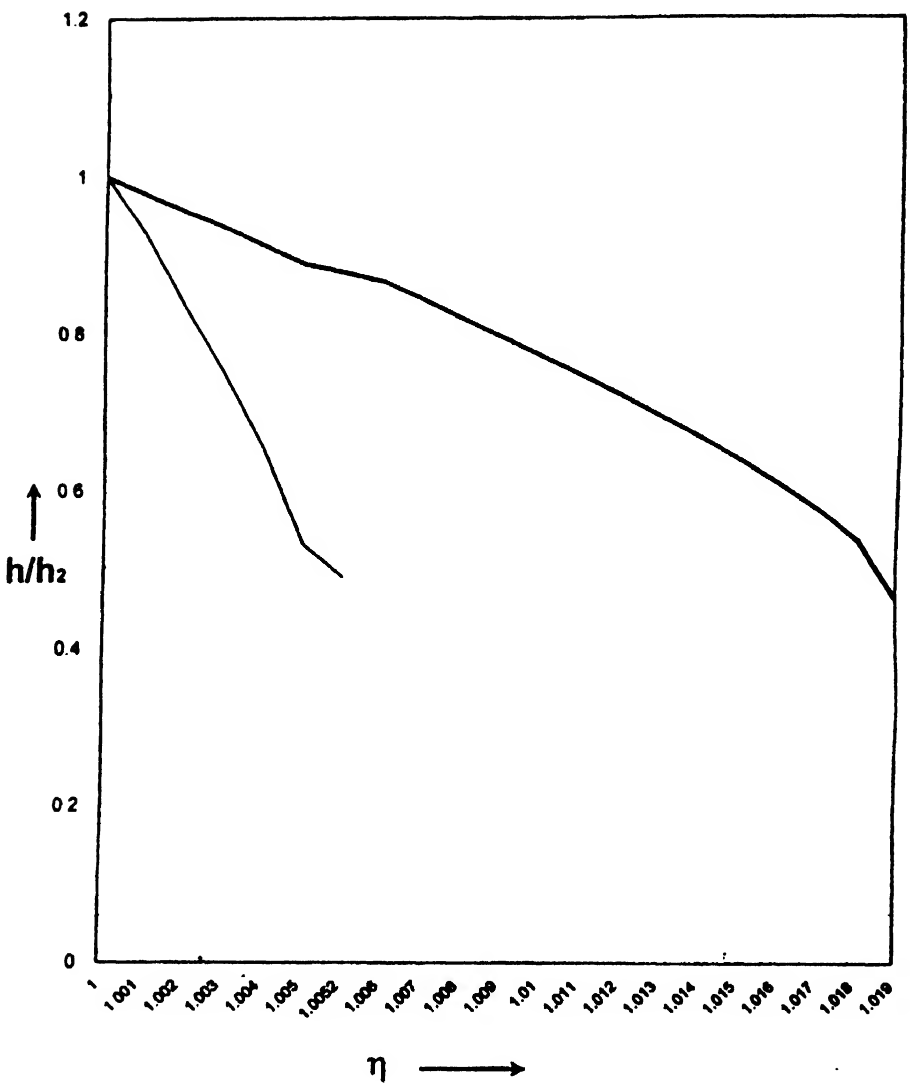

Fig. 5. Distribution of transverse magnetic field

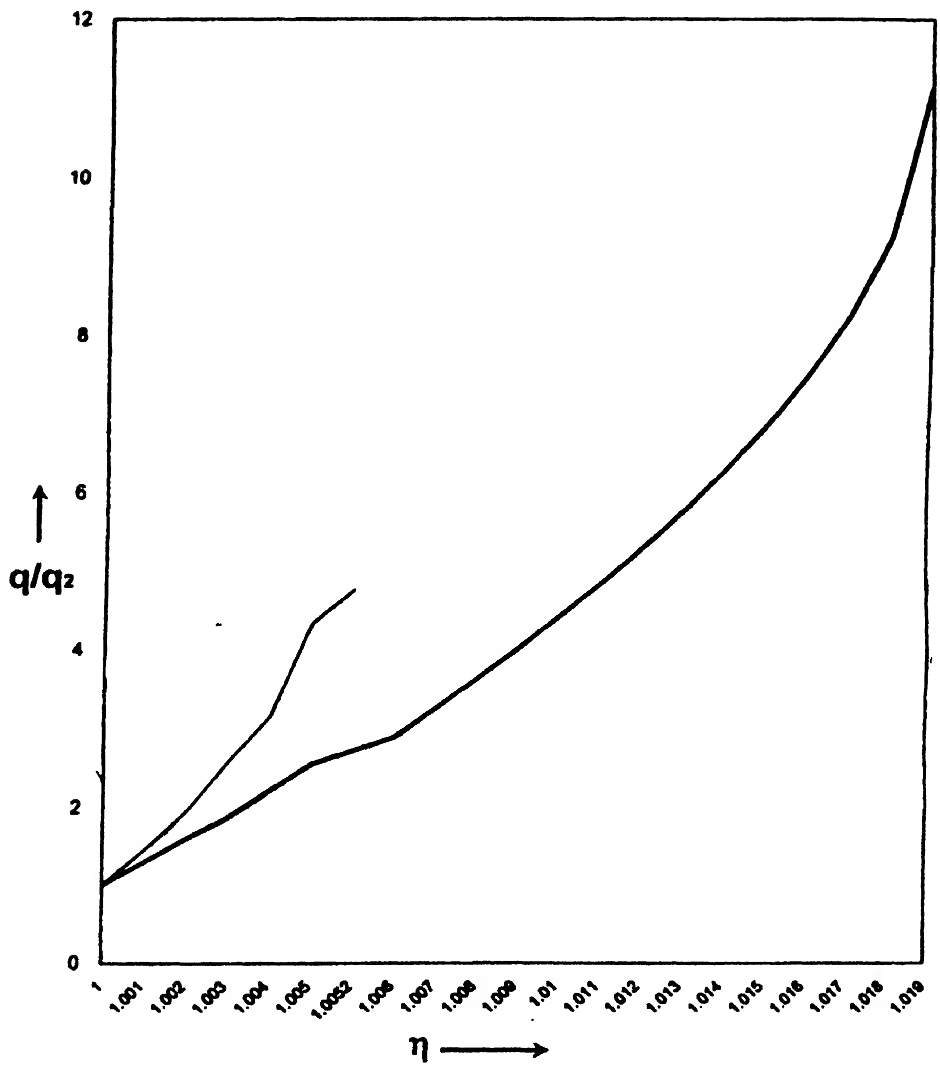

Fig. 6. Distribution of heat flux

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PARALLELS

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ABSTRACT

This article describes the evolution of mathematics in India from a cultural perspective showing how the specific demands of each period in Indian history dictated the kind of mathematics it produced. The title signifies that the development of Indian mathematics paralled the overal culture of this country.

Mathematics Subject Classifications (2000): 01A32, 01A35, 01A85.

OUTLINE

1. **Much Ado About Zero :** Introduces Indian mathematics via its greatest invention the number zero.

2. **Before the Dawn of Time (2500-1700 B. C.) :** The Indus Valley Civilization. Mathematics needed for commerce and architecture.

3. **The CosmicVision (1500-600 B. C.) :** The Vedio period. Mathematics needed for religious rituals; astronomy to determine the proper time of sacrifices and geometry for the shape of the altars.

4. **The Era of System Building (600-300 B. C.) :** The era of great sages (e.g. The Buddha). The birth of combinatorics as a consequence of the philosophical controversies of that period.

5. **The Majesty of the Epic, The Pragmatism of the Polity (300-100 B. C.) :** The era immediately after the Alexandrine invasion. A time of unparalleled cultural development following the unification of India. Zero invented and the decimal system is perfected.

6. **The Supereme Tragedy (100 A.D.-300 A.D.) :** The Gandhara period. Under the stimulus of Greek culture Indian sculpture undergoes a radical transformation and conquers most of Asia. Utilitarianism prevents a similar synthesis in mathematics.

7. **The Golden Age (300-1000 A.D.) :** The Gupta and post Gupta periods. Mathematics in India finally comes of age and becomes a self contained study.

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MUCH ADO ABOUT ZERO

"India's contribution to mathematics is indeed zero". It was to counteract this disparaging and totally unfounded allegation that prompted Balachandra Rao to write a history of Indian mathematics. And yet, on second thought, the above cynical statement is, unitendedly of course, the most flattering compliment that one could possibly pay to Indian mathematics, as it lays bare its most fundamental principle and its greatest discovery, the number zero.

The decimal system, of which zero is an indispensable part, is India's priceless gift to the world. In the numerical value of a digit is determined by its position in a numeral, thus making it unnecessary to have specific symbols for tens, hundreds, thousands, etc. In the Indian system, therefore, just ten symbols suffice to write down any number, while in the Greek and Roman systems an unlimited number of symbols is required. Two other advantages the Indian system enjoys over all others are, first, great compactness in writing down large numbers. It was for this reason that Indians, as early as 1000 B.C., had names for twelve digit numbers, while the largest number given a name by the Greeks was the myriad, a paltry five digit number. And, second, it admits very simple rules, by the almost mechanical application thereof arithmetical operations of arbitrary complexity can be carried out on numbers.

But isn't our numeral system, the reader might object, the gift of the Arabs? It most certainly is, but it must be kept in mind that, in turn, the Arabs got this precious gift from the Indians who were its inventors. There is abundant evidence for this claim. First of all, several Arab mathematicians give this credit to the Indians. Second, the force that pushed the Arabs to the forefront of world affairs was Islam. Before the Hegira in 622 A.D., the Arabs are hardly ever mentioned. It was after that event that they became a first class military power, built one of the biggest empires of all time and eventually developed a culture second to none. This last thing occurred during the second half of the eighth century. And yet, as early as the seventh century, Syrian and Coptic mathematicians began, in tones of awe, to spread the rumour that the Indians were in possession of a wondrous reckoning procedure that surpassed description.

Had Indian genius stopped with the invention of zero and the decimal system, it would have created enormous nonrepayable debt. But it did not. Those amazingly fertile minds created over the millennia glowing and multifaceted gems of mathematical thought, among which are several "firsts" in the history of mathematics.

This article was written to serve three distinct purposes. First, to give a short outline of the history of Indian mathematics. Second, to describe the culture that produced this mathematics. Since originally all mathematics begins it existence by being utilitarian, it faithfully reflects the particular needs of the society that produced it, and every major change in that society's culture is paralleled in the way mathematics is done, hence this article's title. And lastly, to inspire in the hearts of its readers a feeling of reverence for the people who created this culture.

Before we start, a word of caution is in order. Indian history is very sketchy. The flimsy nature of the then existing writing surfaces gave rise to records with little durability. To make things worse, the wholesale destruction of archival material during the middle ages by foreign invaders has led to a scarcity of source documents. Reconstructing Indian history is therefore highly speculative and all proposed dates of specific events must be taken with a large grain of salt.

BEFORE THE DAWN OF TIME

He who writes on ancient history is beset by an unspoken fear as R. Silverberg so eloquently put it. And that is before he can see his book through the press some new archaeological find will refute all his pet theories. That his work will be dated before it even appears!

In 1920 Sir John Marshall started writing the chapter. "The Monuments of Ancient India" for the first volume of the six-volume Cambridge History of India. In it he asserted what, at that time, was the accepted knowledge : that the earliest archaeological finds in India were from the sixth century B.C. When two years later it saw the light of day, it was hopelessly dated as just the previous year the archaeologist's pick had exploded that view. Overnight, the whole pattern of early Indian history was overthrown.

What the pick had brought to the surface were the remnants, in a surprisingly good state of preservation considering their antiquity, of two cities that went back all the way to 2500 B.C.! These were Harappa and Mohenjo-Daro. A hitherto totally unsuspected civilisation had been unearthed. The Indus Valley civilization.

Its carefully planned, well built cities spoke of a possessing a powerful intellect and yet eminently partical. Both cities were built according to strict geometrical planning with parallel avenues intersected by vertical streets; most cities, until very recently, have developed at random. Every house had running water, a bath, and,

wonder of wonders, a perfectly working sewer system. The only other city of the ancient world that could boast of a sewer system as good was the Minoan city of Knossos, modern Europe having developed its own only as late as the eighteenth century A.D.. Finally their granaeries, and other buildings sensitive to moisture, were waterproofed with gypsum of bitumen.

The artifacts found in these cities tell their own story. The people who fashioned them were a literate people who had their own script and who were highly skilled in metal casting, seal engraving, pottery and sculpture. The people who did all this have been called the Harappans and their era (2500-1700BC) the Harappa era.

And how much mathematics did the Harappans know? To build a culture such as the Indus Valley Civilization obviously requires a lot of practical mathematics. But in order to know exactly how much the Harappans knew one must be able to read their script and, so far, this has proved to be impossible. All we can do is hope that one day some brilliant linguist will be able to do for the Indus Valley script what Champollion, Rawlins and Ventris did for the Egyptian hieroglyphic, Mesopotamian cuneiform and Mycenean Linear B. But till that time we must rely on the dumb evidence of the artifacts, and that of two of them is very significant indeed.

The first is an instrument for drawing circles, a sure indication that the Harappans knew basic geometry, a fact evidenced by their very accurate town planning.

The second is a truly remarkable set of weights to be used with their scales. It is the smallest set of weights known in the ancient world capable of weighing the largest number of different objects. This is done by having each subsequent weight weigh exactly double the previous one. These weights taken in order weigh 1, 2, 3, 4, 8,16, 32, ..., etc. units. To weigh objects weighing precisely these amounts is trivial. To weigh an object weighing three units we combine the weights 1 and 2. To weigh objects weighing 5, 6 and 7 units we combine the weights 4 and 1, 4 and 2, 4 and 2 and 1 respectively. Similarly by judicious combination of the first four weights we can weight objects up to 15 units of weight, and using the first five weights up to 31 units of weight. Such a set of weights is the most efficient there is as it achieves its intended purpose by using the least number of weigts. For example, by using the

first 10 weight we can weigh objects as heavy as 1023 units. The reason this works out is that if one adds up all the weights in the above sequence up to any given weight the sum is always twice the final weight minus one. So the next weight in sequence has to be twice the final weight.

The significance of such a set is obvious. Weights are only as good as they are accurate, and cutting accurate weights in the age of bronze was far from an easy task. It was therefore important to have a set with as few weights as possible. The implications for the mathematical maturity of the Harappans is breathtaking. They had discovered the rudiments of both number theory and optimization theory!

THE COSMIC VISION

The Indian soul since time immemorial had a strong affinity for the spiritual and the poetic. It is hardly surprising then that the first expression of religious feeling in India took a distinctly poetic form. They were called the Vedsas. These are collections of hymns addressed to various nature deities, and they also contain instructions on how to appease them and carry favour with by means of sacrifices.

As already stated the dating problem is very acute in Indian history and none more so than in the case of the Vedas. This is so because in addition to every other difficulty already alluded to, the Vedas were committed to writing at a very late date. This entire gigantic opus was intended to be memorized! A second difficulty lies with the subject matter. The Vedas are collections of many poems, and therefore a better name is Vedic Samhitās, which may have been composed over several centuries.

It was in order to settle this vexed question that the celebrated linguist and Indologist Max Müller constructed a brilliant but misunderstood argument by means of which the date of the compilation of the Veads was set at 1400 B.C.. The argument proved nothing of the sort. What it established was that the Vedas could not have been written after 1400 B.C.. Certainly not later than 1400 B.C. but quite possibly sooner.

The Vedas represent one of the first attempts by humans to explore the triple relationship of man and god, man and society and man and his own inner being. The Vedas painted this vast panorama on a truly gigantic canvas.

The cornerstone of Vedic religion was sacrifices which had to be performed in a manner prescribed down to minutest detail by priests called Brāhmaṇas. The

"when", "where" and "how" were all set down in ironclad rules. And it was these rules that gave impetus to the development of three branches of Indian mathematics: Geometry, Astronomy and Arithmetic.

GEOMETRY: The shape of the Vedic altars had to be in regular geometric patterns: square, triangle, circle, rhombus, etc. This and the fact that certain sacrifices had to be performed on two altars each of the same area but different shape, stimulated immensely the development of geometry and also determined its direction. Vedic geometry thus became preoccupied with two problems: Given a certain geometrical shape to find its area, and given that a certain shape was to have a given area to construct it.

A bewildering number of such problems was attacked quite successfully thus giving rise to a vast storehouse of geometrical knowledge. A little after B.C. 1000 this purely ad hoc knowledge began to be organized in texts called Śulva Sūtras, a process which lasted till the seventh century B.C.. If we consider that the father of Greek mathematics, Thales the Milesian, was born in 625 B.C. we can readily appreciate the antiquity of Vedic mathematics.

At the very core of the Śulva Sūtras are two techniques. The first is the theorem of Pythagoras which is so frequently used as to prompt some scholars to call it the Śulva Theorem. It is to be noted, though, that the notion of a proof was as unknown in the Śulva Sūtras as to Pythagoras. The first proof of this theorem was given much later by Euclid.

The second is reductionism. By this we mean the technique of reducing a very difficult problem to one or more easier problems that have already been successfully solved. To appreciate just how much the Śulva Sūtras were ahead of their time it suffices to recall that the method of reduction was used for the very first time in Greece by Hippocrates of Chios (fifth century B. C.) and was systematized by Aristotle (384-322 B. C.).

Before we close this section on Vedic geometry we must mention the most famous problem of the Śulva Sūtras and in fact of all time. Many times the officiating priest had a circular altar and was required to build a square one of the same area and vice versa. This is, of course, the celebrated problem of squaring of the circle which has vexed mathematicians of all persuasions throughout the ages only to be shown to be insoluble by Lindemann in the late nineteenth century. Given this, there are two things about the attempts of Indian mathematicians to solve this problem that

are particularly praiseworthy. The first is the high degree of approximation achieved. The second is the many different attempts that Indian mathematicians made in this direction, a clear indication that they clearly realized the approximate nature of their solutions.

ASTRONOMY : While geometry supplied the "how" of the Vedic sacrifices, astronomy supplied the "when. Since certain sacrifices had to be performed at a certain time, Vedic religion made the possession of an accurate calendar imperative. It was the realization that the regularity of the motion of heavenly bodies could enable one to construct one such that was the beginning of Indian astronomy called Jyotiṣa.

Now of all such motions, two are the most conspicuous of all. The motion of the moon around the earth and the ensuing periodic phenomenon of lunar phases is one. The moon takes approximately one month to go through a complete cycle of phases and the calendar constructed by this means is called lunar. The other is the motion of the earth itself around the sun which, of course, takes about one year to complete, and gave rise to the solar calendar.

It is a well known fact that 12 lunar months do not quite make one solar year. The two calendars are incompatible. The Indians were among the first, if not the very first, to notice this discrepancy and to come up with various ingenious ways to handle the difference.

ALGEBRA, ARITHMETIC & NUMBER THEORY : These are mainly aids to geometry and to astronomy. For example, since the Pythagorean theorem enjoys such preeminence in the Śulva Sūtras, it should come as no surprise that Vedic algebra and arithmetic are heavily involved in solving quadratic equations and finding the square roots of numbers which are not perfect squares, respectively. And yet even at this early time, for some inexplicable reason, Indian mathematicins became involved with one of the most difficult problems of that branch of higher mathematics called number theory. Number theory concerns itself with integers exclusively. To the uninitiated this may sound rather prosaic; after all, problems on integers must be pretty tame. In reality, however, these problems, far from being tame, are nothing short of horrendous.

Now the particular problem that made its appearance in the Śulva Sūtras belongs to that branch of number theory called indeterminate analysis. It has to do with equations of more than one unknown. While equations in one unknown have single solution, many unknowns have several solutions, in fact infinitely many. For example,

the equations $2x + 3y = 7$ and $2x + 4y = 11$ both have infinitely many solutions, since for any value of x, y receives a corresponding value such that the equation is satisfied. But suppose we require that these solutions satisfy some extra conditions, the one most frequent is that these solutions must be integers; i.e., fractional values are excluded. Another condition is that, in addition to being integers, these solutions must be positive numbers, Then a strange thing happens, as a moment's reflection will convince that the first equation above has integer solutions, $x = 2$ and $y = 1$ is one such, but the second equation does not. That is to say whenever x is an integer, y can take only a fractional value. (Does the reader see the why of these two assertions?) The most famous such problem is the so called problem of Fermat (17 century A.D.) which was finally solved in the last decade of the 20th century, by Andrew Wiles and whose proposed solution is 160 pages long!

All this raises a very reasonable question. The whole reason of Vedic mathematics is a highly utilitarian one. So what practical purpose, if any, is served by considering a problem in such an esoteric field as indeterminate analysis. I have no easy answer to offer since the documentary evidence offers no clue on this subject. I can make a conjecture, though, which I think gives an adequate answer.

As stated, Vedic sacrifices had to be offered at certain times. There were some to be offered at the end of a month, and others at the end of a year. But what if the end of a certain year actually coincided with a month's end? Such a coincidence might seriously affect the ritual of these two sacrifices. But as we know such a coincidence is rather rare, the solar year being a little more than 12 lunar months. But such years that are coterminus with the end of a month do occur. The problem faced by the mathematician was to predict them. Indeterminate analysis is the tool to use for this end. If we assume rather simplistically. That one year is 365 days and each month is 30 days then the problem reduces to finding all the positive integer solutions to the equation $30x + 365y = z$.

What I just said is highly speculative, but I hope it may inspire some mathematician well versed in the archaic form of Sanskrit used in the Vedas to do the required research and see if it bears me out.

THE ERA OF SYSTEM BUILDING

The sixth century B.C. is a unique landmark, one never equalled before or after. It marks a radical change in the way men viewed themselves in the cosmos. In the sixth century man's thinking made an irreversible transition from the mythological

mode to the analytical. It is a time when great sages began teaching their individualistic and idiosyncratic messages regarding problems that remain unfathomable to this very day.

And there was hardly a part of the world that did not experience this phenomenon some way or another. The sixth century saw the pre-Socratic philosophers in Greece, Zoroaster in Persia and Confucius and Lao Tzu in China. These, to quote Aldous Huxley, are enormous names, inaccessible eminences. And India had the lion's share of the crop, as many of the questions asked at that tie already existed, albeit in embryonic form, in the Vedas, and she was now ready to take the next step. And the next step involved a bifurcation of the Vedic message into the antithetical directions of mysticism and philosophy.

But what was so unique about the sixth century as to bring forth all these changes? In one word it was profound sense of dissatisfaction that was keenly felt by the more sensitive, coupled who the courage to start a new. And in India this dissatisfaction was aimed at the excessive ritualism of the Vedic religion and, by association, at the highly organized, top heavy, priesthood that performed these rituals.

A reaction was inevitable and it took two distinct forms. The first was an out and out revolution. Three deep thinkers developed their own particular systems which deny the efficacy of ritual and sacrifice, and reject the authority of the priests and that of the Vedas. These were Cārvāka who developed the Lokāyata system, Mahāvīra who developed Jainism and Gautama who developed Buddhism. The first has disappeared; the Jains are now a rather small community in India; but Buddhism spread like wildfire, influenced Indian thought in all its aspects and finally conquered much of Southeast Asia and the Far East.

The fact that both Gautama and Mahāvīra were members not of the priestly caste of Brāhmaṇas, but the warrior caste of Kṣatrīyas, plus the fact that they totally rejected both the caste system and its ensuing pretension that some people are born with a virtue inherited from their parents, underline the strong reaction to priest craft that is the main characteristic of the sixth century. On a deeper level, though a different picture emerges as the entire conceptual framework of both Buddism and Jainism is borrowed wholesale from the Vedas.

It was this realization, that the Vedas admit of more than one interpretation, that prompted many Indians to adopt an evolutionary approach to the Vedas rather than revolutionary one advocated by Cārvāka, Mahāvīra and Gautama. By the time these three, heterodox systems, as they became known, were developing, there was

already afoot a rival approach whose aim was not to reject the authority of the Vedas but to lay bare their essence and to restore to religion the ancient purity robbed by Brāhmaṇical sacerdotalism. The fruit of this approach was a series of philosophical-mystical treatises called Upaniṣadas.

The Upaniṣadas reaffirm the authority of the Vedas, and at the same time reject much of the sterile ritualism that grew around them. Thus the Chandogya Upaniṣadas tells the story of a Brāhmaṇa's son called Svetaketu who left his father's house and went to study for the priesthood and after twelve years he returned home, very much puffed up with his own learning. This until his father asked him if, in addition to all the knowledge he had acquired, he had found out who the knower of this knowledge was. Or in other words "know thyself". From the same source comes the story of one Satyakāma, who was the illegitimate son of an outcast woman and yet was accepted for religious instruction by a sage when the latter realized Satyakāma's great potential.

These works of Vedic literature, from the Samhitas all the way to the Upaniṣadas, became the scriptures of the so called orthodox schools, which in turn have survived under the generic name of Hinduism.

After this lengthy preamble we must have a look at the mathematical innovations of this period. It will of necessity be a very brief look as our knowledge of Indian mathematics at this period is almost nil. The ravages of time have obliterated practically all documentary evidence. By comparison, the Vedic period with the Śulva Sūtras and the Jyotiṣa rich in source documents. The little we do know has been gleaned from the philosophical texts of this period. And among these gleanings there are two significant for mathematics.

First, as already stated, by 1000 B.C. the Indians could call twelve digit numbers. This skill was enormously developed till in the sixth century numbers with as many as 53 digits could be called. This we know from the mouth of Gautama who had received an outstanding education and prior to his entering the ascetic life he was fond of taking part in mathematical competitions, which were quite popular at that time. And the fact that such competitions were being held says a lot of the cultural level in sixth century India.

The second innovation, by far the most important, goes a very long way to show the unbelievable mathematical maturity of the Indians, as well as their philosophical sophistication. It was the complex doctrines of the Jains that gave rise to this particular development. Like all philosophical systems, Jainism proposed

certain basic principles that were thought to be fundamental to the system, and all else followed in strict logical sequence from these principles. A natural question came; what would happen if one or more of these principles was modified or even dropped altogether? Or more generally, by assuming some but not all of Mahāvīra's assumptions, how many distinct systems could one produce? It was from this most unlikely beginning that a mighty new branch of mathematics was born : combinatorics. This particular discipline occupies itself with such questions as, how many ways can we rearrange a give number of objects and, given a set of objects, how many distinct subsets can we form from these objects such that each subset contains fewer objects than the original set? The study of combinatorics is relatively a newcomer in the west; it was initiated in the 18th century A.D. by the genius of Leonard Euler. In India, by contrast, it got started by about 500 B.C.

THE MAJESTY OF THE EPIC, THE PRAGMATISM OF THE POLITY

Like a fiery meteor he came from the west astride his magnificent steed. He struck without warning like lightning from a clear sky and with the same devastating effects, he moved with unheard of speed and attacked before his enemies had a chance to field an army. And when they did it availed them but little against his brilliant tactics. In three textbook battles he routed the formidable but ill led Persian armies. He stormed supposed impregnable fortresses, some situated on islands, and some on mountain tops. And having demonstrated his mastery of regular warfare, chameleon like he adjusted his tactics to the irregular warfare of the nomads of central Asia. His name was Alexander; he was dead at 33 years of age and yet he had already earned his place in history as the Great.

Having conquered the Persian empire and the steppes of Central Asia, in 327 B.C., he invaded India. On the banks of the river Hydaspes (Jhelum) he gave the last of his major battles. The Indian army under King Porus (Paurava) was annihilated. Once more Alexander had won the battle, but this time he had lost the war. The Greeks were so impressed by the Indian army, and not least of all by its formidable war elephants, that now, after having followed Alexander half the way around the world, exhausted in body and weary in mind, and on the brink of mutiny, they demanded that they be allowed to return home. Alexander was forced to call the invasion off, returned westward and died in Babylon. The full and just measure of his short life is the pithy statement of Sir William Tarn : "Nothing could be again as it had been".

The danger to India was not yet past, because twenty years later, Seleucus Nikator, who was one of Alexander's ablest generals and the inheritor of the Asiatic part of his empire, was now poised for a new invasion. And the timing could not have been worse, for even at the best of times India had been nothing more than a patchwork of rival principalities, lacking unity. Now. Alexander's short lived but devastating invasion had engendered a moral confusion, despair, and despondency, a sickness unto death. A renewed invasion could have but one outcome.

Deliverance came in the person of one man, Candragupta Maurya. He took advantage of the already existing political vaccum to size power and make himself king. And not a second too soon as the Greek invasion was now underway.

One can hardly imagine a more unequal contest. Seleucus Nikator was a proud member of the Macedonian military aristocracy and his authority was never disputed. Candragupta Maurya, on the other hand, was the son of a herdsman, and this constituted a formidable social infirmity in caste conscious India. The morale of the Greek army was never higher it having never suffered a defeat, while that of the Indian was at a low ebb, the defeat at Hydaspes still being fresh. And this was not all. The army of Seleucus was even deadlier than that of Alexander, because the former, in addition to its traditional components of heavy infantry and heavy cavalry, had augmented it with light cavalry drafted from among the superb horsemen of central Asia, and even an elephant corps from what today is Pakistan. Finally Candragupta was an untried stripling. Seleucus, by contrast, was a battlehardened veteran who had served his apprenticeship at the feet of the greatest military genius of all time.

Nobody entertained any doubts about the outcome when Candragupta Mauraya took the field against Seleucus Nikator. And yet, when the dust had finally settled, the Indian standard was flying victorious. The only thing more amazing than this victory of Candragupta is the shabby treatment he has suffered at the pens of western military historians in whose books he is not mentioned even in a footnote.

Having prevailed on the filed Candragupta went one better with a brilliant diplomatic coup with which he not only restored peace but even made a lifelong friend of Seleucus. He then proceeded to conquer most of India save the extreme south, and thus created the first Indian empire and gave his name to the dynasty that ruled it, the Maurya dynasty. And he gave to India something else of incalculable value, a national identity.

Empires are conquered by the sword but are ruled by the pen. Candragupta

once more rose to the occasion when he picked Kautilya as his prime minister. The latter created the most absolute royal autocracy imaginable, one ruled by carefully codified laws and administered by a highly centralized civil service. And it is to the per of Kautilya that we owe one of the first treatises on statecraft ever written, the Arthaśāstra.

Candragupta brought about a military-political revolution, his grandson Aśoka a philosophical and social one. And it started when the latter converted to Buddhism. Scholars have tried to explain his conversion in sundry ways. One view holds that the war of Kalinga caused in him an aversion to violence which made him sympathetic to the tenets of this pacific faith. Another maintains that it was due to the influence of his wife, herself a devout Buddist. A third view has been proposed by Romila Thapar. According to this, Aśoka embraced Buddhism in order to enhance the middle class (Vaishyas) at the expense of the priestly and warrior castes. Aśoka himself a Maurya and hence neither a priest nor a warrior always suspected these two castes of potential sedition. His primary motive though was an economic one as the Maurya age was one of unprecedented economic opportunities and demands. The, by now, huge civil service had to be paid, and the treaty of friendship with the Seleucids opened enormous opportunities for commerce. Hence the need to strengthen the Vaishyas who were the income generating class.

Whatever the reason for his conversion, its effects are inestimable. Once Buddhism became in effect the state religion, it grew by leaps and bounds and, while in India it was destined to eventually disappear, it had gathered sufficient momentum to cross the border into Tibet, Southeast Asia and the Far East where it took root.

India under the vigorous rule of the Mauryas experienced enormous wealth and also feelings of national pride, security and optimism. It is hardly surprising, therefore, that this period saw a cultural renaissance. When the Indus Valley Civilisation came to an end by 1700 B.C. the visual arts almost disappeared. The reign of Aśoka ushered in a veritable explosion in the art of sculpture, which experienced a rare stage of technical perfection. One device in particular became a convenient way for dating sculpture. The sculptors of this era developed a secret technique one subsequently lost, of giving sculpture a characteristic polish. This allows to place statues within this period and is known as the "Mauryan gloss". The capital of a column found near Sarnath has become in our days the national cemblem of India.

Since Vedic times bards sang the tribulations and triumphs of mythological heroes. It was, however, in the time of the Mauryas that poets of genius collected these isolated songs and combined them to create the two longest epice poems ever written, the Rāmāyaṇa and the Mahābhārata,. There are three things noteworthy of these poems. The first is the majestic style in which they were written, one that reminds one of the Vedas and seems to be a reaction to the dry over analytical style of the Upaniṣadas and the Buddhist Sitars. The second is that the epics have proved to be an invaluable source of information about life and mores in Vedic times. And the third is a new doctrine found in the Bhāgavad Gītā which is a philosophical appendage to the Mahābhārata. And this is that the righteous man or woman who does not abandon society is as good as the ascetic and the sage. That the life of action can be as meritiorious as the contemplative life. A tribute to Mauraya pragmatism!

But the greatest single achievement of this period, and in fact of all times, was the invention of zero by the mathematician Pingala. Its significance has been sufficiently discussed in the introduction to make further comment redundant. However, one question will be discussed here, now that we know something about Buddhism. And that is why zero originated in India and not, say, Greece. The answer is that Greek arithmetic remained associated with geometry and therefore had to measure some positive quantity, be it length, area, volume or whatever, and zero does not measure anything. By contrast, one the most basic ideas of Buddhism is that of emptiness. An emptiness which is the very apotheosis of nothing and yet has the enormous potential to create everything. And in mathematics this nothingness and its tremendous potentiality is exactly the number zero.

THE SUPREME TRAGEDY

Many years ago while on a trip to Toronto I visited one of its museums. I walked by, taking in the exhibits when all of a sudden I stopped dead on my tracks in front of a statue of the Buddha, my eyes wide with excitement and hardly able to breathe.

To me it was as if the Buddha through his enigmatic half smile was trying to tell me something about the artist who had fashioned him. The heart and soul of the anonymous artist were undoubtedly Indian, but the hands that had given shape and form to the cold stone were the hands of a Greek. It was my first encounter with Gandhara art, my first forestaste of its treasures.

So let us look briefly at the events that brought this amazing, if temporary, marriage between two of the greatest civilization of all time, and whose offspring was that particular Buddha I had so admired.

The death of Aśoka signalled the decline and eventual dissolution of the Maurya dynasty. The very authoritarian autocracy that Kautilya had created, and which in capable hands had proved to be so effective in guiding the ship of the state, in the hands of the mediocre kings who succeed Aśoka proved to be the destruction of the state. The monolithic regime of Kautilya totally lacked the checks and balances so essential for a healthy state of affairs.

First the Deccan Plateau separated from north India rule under its own dynasty, the Andhra dynasty. Then central Asia once again experienced a turbulent era of nomadic invasions which further weakened India. And it was at that time that she experienced one last wave of Greek invasion. In the extreme east of the old persian empire, called Bactria, descendants of Alexander the Great had gained complete autonomy from the moribund Seleucid dynasty, established themselves as rulers of that part of central Asia and embarked on a rigorous expansionist policy eastwards. Three of the Greco-Bactrian kings as they became known, Euthydemus, Demetrius and Menander established a short lived rule over today's Agghanistan, North Pakistan and the Punjab. They proved to be not only fierce warriors but also wise administrators and, which is most important for our own purposes, sensitive artistic men.

A great many of the coins issued by the Greco-Bactrian kings have survivedm mute witness to the great affluence this region enjoyed under their rule. These conis, which are considered artistic masterpieces by experts in numismatics, display a fascinating dualism. On the one side these coins depict a Greek theme, usually the king's profile with his name in Greek letters, but on the reverse side the theme is almost invariably Indian. The hybrid nature of these coins represent faithfully the overall culture of this era.

In the Buddhist canon there is one book titled "Questions of king Milinda". It consists of the questions of a king, newly converted to Buddhism and the answers given him by his spiritual guide, the monk Nāgasena. That king was none other than the Greco-Bactrian king Menander, about Lwhom we have from independent sources that he had embraced Buddhism, and Milinda is simply the Indianized version of his name.

Under the royal patronage of Menander, Buddhism invaded Afghanistan and

Central Asia where it became the state religion till it was supplanted by Islam. And it was because of the influence of Greek aesthetic ideas that Buddhist iconography and, by extension, the whole of Indian art underwent a remarkable metamorphosis.

Unbelievable as it may seem to the uninitiated, both Hinduism and Buddhism were originally radically iconoclastic as the most cursory reading of the Vedas and the Buddhist Sūtras will indicate. Now in the Maurya period a lot of high quality Buddhist sculpture was produced, still the actual form of the Buddha was never shown. Instead his presence was indicated by symbols. For example, a tree would represent the actual tree under whose shade Gautama experienced his Buddhahood. A wheel would represent his doctrine that he set in motion. The closest to an actual representation of the Buddha that was ever made by Maurya artists was to show his footprints.

The Greek spirit with its reverence of the human body and its delight in the material world changed all that, and indeed it sounded the death knell of the excessive other worldlines of Buddhism. From now on, the image of the Buddha, and a little later those of the Hindu deities, will unapologetically take centre stage. Everyone recognizes the contribution of Greek artists in depicting the naked human body in all its glory. What few people realize, though, is that their contribution in depicting the clothed body are of an even higher order. And Indian artists took to draperies with a vengeance. Long after Buddhism disappeared in India, and Greek influence was but a forgotten memory, Indian artists continued to produce sumptuously draped statues.

Now Ghandhara never became part of the Greco-Bactrian empire, but the Indian king Kanișhka invited Greek artists in his court there and the city became the cultural centre of India at that time, and its name became the name of the Greco-Indian style of art. And it was the Gandhara style of sculpture rather than the Maurya style that emigrated first to China and then to the rest of the Far East and took those parts of the world by storm. So imagine! The giant Buddha of Kamakura in Japan actually has some Greek blood!

Far less known than the sculpture of this era is a contemporary mathematics manual which came to light, in a sad state of decomposition, in 1881, near the village of Bakshali, and hence came to be known as the Bakshali manuscript, and is now reverently kept in the Asmolean library at Oxford. In the Bakshali manuscript the decimal system based on the first nine digits and one extra symbol to represent zero, reigns supreme. Another proof that the Arabian numerals are really Indian.

The other striking feature of the Bakshali manuscript is the deadening spirit of utilitarianism that reigns throughout, and which may go a long way in providing an explanation for the why of what is, in my mind at any rate, the greatest cultural misfortune of all times.

The two mathematical systems of the ancient world that achieved the highest degrees of subtlety and depth were those of the Greeks and the Indians. Each had exactly what the other lacked. Indian mathematics had zero and the ensuing decimal system. The Greeks did not have it, something that made the performance of simple arithmetical calculations a horrendous: task. Consider the two last and btightest lights of ancient Greek mathematics: Applollonius of Perga and Archimedes. Their work includes several breathtaking firsts: On conic sections, on spirals, on the integral calculus (Yes! It goes that far back!), on optics, on statics, on fluid mechanics; the list is endless. And yet concerning these two giants, the distinguished historian of Greek mathematics, Sir Thomas L. Heath, wrote as follows: "With Archimedes and Apollonius Greek geometry reached its culmination, indeed, without some more elastic notation and machinery such as algebra provides, geometry was at the end of its resources."

On the other hand what Greek mathematics had, and its Indian counterpart lacked, was the axiomatic method. By that we mean a very orderly and highly structured method, in which, first the objects of one's study are defined, then certain self-evident truths about these objects, called axioms, are set down and then all further claims about these objects, called theorems, are proved using only the axioms and the theorems already established.

The Greeks in the geometry of Euclid, "The Elements," possessed just a system. A masterpiece of abstract thinking so deep that after all that time it can still cause the infirm to have nightmares. And it is the absence of such and axiomatic system that is the greatest defect of Indian mathematics. When one examines the work of Indian mathematicians, even the best organized like Līlavatī of Bhāskarācārya, one is struck by the fact that these works are nothing more than adhoc collections of problem solving methods without deep unity.

And one is forced to put forth the question, what prevented these two great mathematical systems to merge into one system enjoying the best of both and suffering from none of their weaknesses? As we saw when we looked at the Gandhara style of art, such a course, though difficult, was not impossible. But it never happened and

this is the supreme tragedy in the history of ideas. Had it happened, what possibelities would be available? We can not imagine. All we can do is duly note the fact and weep.

7. THE GOLDEN AGE

Western historians have traditionally desginated the period from roughly the 4th century to the 11th century of the common era as "The Dark Ages", Strictly speaking this lugubrious sobriquet applies only to western and central Europe. There were other parts of the world where civilization achieved rare heights of vitality and refinenent; something that same westerners find rathe disconcerting.
To illustrate with examples, the Byzantine Empire experienced during this period not one but two distinct golden ages, one under emperor Justinian, and one under the Macedonian Dynasty. The Muslim world reached a unique cultural zenith under the Abbasid Caliphate in Baghdad. The apogee of Chinese culture occurred under the mighty Tang Dyanasty. And India knew its greatest splendour under the Gupta Dynasty (320-550 A.D.) and its immediate successors, a splendour that lasted well into the eleventh century. And, incidentally, the unequalled culture of the Tang Dynasty owes a lot to its contemporary India as at that time so many elements of Indic culture entered China.

The main cultural elements of that time are as follows : Buddhism in India has entered its long decline and eventual demise, and a vigorous Hinduism is now reasserting itself; abroad Buddhism is spreading like wildfire, but at home its days are numbered. Hinduism, however, for all its vitality, or perhaps because of it, is experiecncing fragmentation. Already centuries before, deep thinkers were interpreting the Vedas and the Upaniṣadas differently. And by the time of the Gupta Dynasty these different interpretations finally crystallized in the six major philosophical systems of Hindusim.

In literature one name eclipses all others, that of Kālidāsa. Quite apart from his literary merit, and that is enormous, under his pen Sanskrit achieved its highest state of perfection. In fact what, much later Dante, Alighieri did for Italian, Chaucer did for middle English and Martin Luther did for high German, Kālidāsa did for Sanskrit.

The visual arts experienced in this period growth that is nothing less than explosive. Sculpture having absorbed from the Greeks of Gandhara whatever was

useful for its needs, now became once more a thoroughly Indian art, no longer looking outside for inspiration and technique, but goging inwardly to draw on its own life force. And Buddhism that for such a long time had dominated the artistic scene, now has to share the limelight and eventually to be eclipsed by Hinduism whose deities are now claiming the attention of the artist.

While on the subject of the visual arts, there are two innovations of that period which changed completely the face of sculpture and also that of architecture. While in previous periods the material of choice for statues were sandstone mainly, but also schist, now sculptors and architects started working with soapstone. This remarkable rock is quite soft when freshly quarried and therefore easily carved into statues and reliefs. But prolonged exposure to air hardens it so that anything already carved on it becomes indelible. This allowed the construction of huge temples whose entire wall surfaces are covered by tracery and reliefs of great complexity and fineness. It was in fact during this era that the Hindu temple acquired, with many variations of course, its classical form.

The second innovation was the use of the lost-wax process to cast bronze statues. This process works roughly as follows: The image of whatever is to be made is first modelled in wax; wax being very malleable, this allows for extraordinary fine details to be made. Then the wax is covered with clay which is left to harden. Afterwards the entire thing is heated until the melted wax runs out through specially constructed openings, thus leaving inside the clay a cavity which exactly duplicates the original wax figure, Then molten bronze is poured in and allowed to cool. Finally the clay is broken away and what is left is the complete bronze image. The sculptors of South India became top masters in working with this technique.

Personally, there have been two cast bronze statues that I have admired best of all. The one is the world famous Perseus by Benvenuto Celini in Florence. The other, not quit Ee as well known, quite obscure realy, is a magnificent Shiva Nataraj at the Musée Guimet in Paris, which must be the finest museum of oriental antiquities in the world. And as far as artistic inspiration goes, or virtuosity of execution, there is not much of the one that one may wish on the other.

As already stated, while Greek art made great inroads in India during the Gandhara period, Greek mathematics failed to do so, and by the time of the Gupta period all relations with the Hellenistic word had been severed so no exchange of ideas was now possible. This is particularly galling because the best mathematicians that India ever produced, lived and laboured in the Gupta and post Gupta periods.

And here we come to a decisive turning point in the history of Indian mathematics. The mathematicians just mentioned produced such breakthrough research that they more than pushed the limits of mathematical knowledge; they changed the very nature of mathematical practice in India. Till the Gupta period, all mathematics had been, in some way or other, utilitarian. It always had to do with when to offer sacrifices, or the shape of altars,r the number of distinct Jain philosophies one could produce from Mahāvīra's doctrines, or with problems of commerce, or metallurgy and the like. The uses it was put to varied with each passing era, but it always had to serve some practical purpose or other; it had to serve the needs of society as those needs were being perceived in each era. And this has been the central theme of our article: mathematics paralleling the culture of India. Now, however, mathematics gains its independence from society's needs and becomes a study undertaken for its own sake. Mathematics is still applied to astronomy; nearly all mathematicians of this period were also superb astronomers, but astronomy itself also gained its independence from parctical considerations at this time. Ane therefore mathematics stopped paralleling the society that produced it, and so with a heavy heart I must now retire from my project and take my leave from my readers.

And yet I do not wish to leave my topic without a brief mention of at least a few of them. Seeing that the third reason for writing this article is to inspire a spirit of reverence for the achievements of the Indian mind, this can hardly be better served than by taking a look at four of the mathematicians of this period and noting their major achievements, and also their "firsts". That is, innovations that they made and were independently rediscovered in the West centuries later. The mathematician reading these "firsts" who fails to experiecnce a tingle up his spine is a very jaded man indeed.

Āryabhaṭa (b. 476 A.D.) : He started it all. He found mathematics in decline and revitalized it. Many old ideas had been lost and he was forced to reinvent them before he could push on, a fact that has earned him the inaccurate title of father of Indian mathematics. His work was on a unique technique for calling enormous numbers, one based on the Sanskrit alphabet, the geometry of triangles, on the squaring of the circle, indeterminate analysis, trigonometric table, to propose the helioentric theory, and to use geometric techniques to solve algebraic problems.

Brahmagupta (b. 598 A.D.) : He took over where Āryabhaṭa had left off, and enormously advanced his pioneering work. In trigonometry he invernted the second order interpolation, fully one millennium before Newton and Stirling, for finding

the sine of an angle whose value lies between that of two angles whose sines are known. In indeterminate analysis he was the first to give a method, albeit without proof, for solving the so called Pell's equation a thousand years before John Pell (1611-1685).

On the negative side, Brahmagupta was arrogant, vainglorious and something of a sadist. His favourite pastime was discovering and gleefully exposing the errors of mathematicians past. The Jyotiṣa, the Śulva Sūtras, Jain mathematicians - nothing was too sacred and nobody was safe from his invective. But his most vitriolic abuse and his choicest epithets he reserved for Āryabhaṭa for whom he harboured an almost pathological hatred. And while this is very unfair, of course, I still find his irreverent attitude towards authority most refreshing. In India where the prevailing feeling has always been "the Guru can do no wrong", Brahmagupta cuts a delightful figure.

Mahāvīrācārya (9th century) : As his name implies, he was a Jain. It is therefore not surprising that, while he contributed to all branches of mathematics, his main innovations are in the field of combinatorics where he generalized known results for a small number of objects to arbitrary many objects.

Bhāskarācārya (b. 1114 A.D.) : With him we reach the apex of Indian mathematics.

After his death the subject enters its decline and with exception of Kerala where some brilliant astronomers laboured, original mathematical research dries up. He worked in many fields, but his forte was indeterminate analysis and astronomy. In indeterminate analysis he advanced by leaps and bounds Brahmagupta's work and invented methods superior to those invented by Lagrange in the eighteenth century. In astronomy, his work on the motion of the moon preceded by four centuries the work of the Danish astronomer Tycho Brahe. And it is in astronomy that he made his greatest discovery. He reasoned that when the instantaneous velocity of a planet attains its maximum, its acceleration must vanish, otherwise the velocity would increase some more and thus violate the assumption that it had reached its maximum. Working on this and the fact that the periodic motion of planets is best described by trigonometeic functions, Bhāskarācārya found a method for differentiating such functions and setting them to zero so that the instantaneous acceleration would vanish. He had discovered, in other words, the differential calculus five hundred years before Fermat.

Having come so far we have reached the maximum in our endeavours and thus can go on no further, something that Bhāskarācārya would understand having himself said it a long time ago. It is therefore only fitting that with this magnificent example we should wish our weary readers a very fond of God speed.

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A MODEL OF MAGNETOGASDYNAMIC CYLINDRICAL SHOCK WAVE IN UNIFORM NON-IDEAL GAS

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ABSTRACT

Self-similar solutions have been investigated for one-dimensional unsteady flow of a non-ideal gas behind a cylindrical shock wave in the presence of transverse magnetic field. The total energy of the wave is increasing as some power of shock radius. The numerical solutions are obtaind for various values of the parameter that characterises the non-ideal nature of the gas.

Key Words : Non-ideal gas, shock wave, Magnetic filed.

INTRODUCTION

The theory of shock waves and related flows are of immense importance in the theory of sonic booms, phenomena associated with LASER production of plasmas, high altitude nuclear detonation, supernova explosions and sudden expansion of corona into the interplanetory space.

The self-similar problem in the non-uniform ideal gas was studied by Pai [1], Christer and Helliwell [2], Ranga Rao and Ramana [3], Singh and Vishwakarma [4] and many others. But when the flows take place at high temperature the assumption that the gas is ideal is no more valid. Recently Singh and Shrivastava [5], Singh [6] have discussed the shock wave in non-ideal gas on the basis of paper Ranga Rao and Purohit [7].

Almost ideal gases such as low density gases are specified by an equation of state different from the ideal gas.

In the present problem we have studied self-similar solutions of the flow behind the cylindrical shock wave in uniform non-ideal gas taking into account the transverse magnetic field. Total energy of the flow between the shock front and

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contact surface is assumed to be determined on shock radius obeying the power law. The results are obtained for different values of the parameter alpha (α) which characterises the non-ideal nature of the gas. Results are illustrated through tables.

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Here we start with equation of state taken from the paper of Ranga Rao and Purohit [7] and Anisimov and Spiner [8],

$$p = \Gamma \rho T(1 + b\rho), \tag{1}$$

where b is internal volume of the molecules and Γ is a gas constant. The initial magnetic field distribution is assumed to be

$$H_0 = Ar^{-n}, \qquad (n>0) \tag{2}$$

and the total energy of the wave is taken as

$$E = Br^{m}, \qquad (m>0) \tag{3}$$

where A, B, n and m are constants. R is the shock radius.

The initial density is

$$\rho_0 = constant \tag{4}$$

The differential equations governing the motion of the polytropic gas under the influence of magnetic filed are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u) + \frac{\rho u}{r} = 0, \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu H}{\rho r}\frac{\partial}{\partial r}(rH) = 0, \tag{6}$$

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial r}(uH) = 0, \tag{7}$$

$$\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial r} + p\left[\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + u\frac{\partial}{\partial r}\left(\frac{1}{\rho}\right)\right] = 0, \tag{8}$$

where we assumed that

$$e = \frac{p}{(\gamma-1)\rho(1+b\rho)}; \tag{9}$$

implying

$$C_p - C_v = \frac{T\left(\frac{\partial p}{\partial T}\right)_v}{\left(\frac{\partial T}{\partial v}\right)_p} = \Gamma, \tag{10}$$

a result which follows from equation (1) provided the term $b^2\rho^2$ is ignored. The symbols ρ, u, p, H, e denote density, velocity, pressure, transverse magnetic field and internal energy respectively.

The jump conditions to those of strong shock are

$$u_1 = (1-\beta)V, \tag{11}$$

$$p_1 = (1-\beta)\rho_0 V^2, \tag{12}$$

$$\rho_1 = \frac{\rho_0}{\beta}, \tag{13}$$

$$H_1 = \frac{H_0}{\beta}, \tag{14}$$

where $\beta(0<\beta<1)$; is obtain by the relation

$$\beta^3 M_A^2(\gamma+1) - \beta^2\left[\gamma + M_A^2(\gamma-1+\alpha)\right] + \beta\left(\gamma - 2 - \alpha M_A^2\right) + 2\alpha = 0; \tag{15}$$

with

$$\alpha = b\rho_0(\gamma-1); \tag{16}$$

and V denotes the shock velocity. For a given set of values of α, γ and M_A the value of β can be obtained by solving the equation (15), in which M_A is known as Alfven Mach number and is given by

$$M_A = \left(\frac{V\rho_0^{1/2}}{\sqrt{\mu}H_0}\right), \tag{17}$$

and suffix 0 and 1 denote the values just ahead and behind the shock front respectively.

SIMILARITY SOLUTIONS

Let us suppose

$$R = Ct^{\delta}, \tag{18}$$

where C is constant and

$$\delta = \frac{2+m}{4} = \frac{1}{1+n}; \tag{19}$$

obtained by dimensional analysis of Sedov (1982).

We introduce the following similarity transformations for finding out exact numerical solutions,

$$\left.\begin{aligned} \lambda &= \frac{r}{R} = \frac{r}{Ct^{\delta}}, \\ u &= \frac{r}{t}U(\lambda), \\ \rho &= \rho_0\Omega(\lambda), \\ p &= \rho_0\frac{r^2}{t^2}P(\lambda), \\ \sqrt{\mu}H &= \sqrt{\rho_0}\frac{r}{t}K(\lambda), \end{aligned}\right\} \tag{20}$$

where U, Ω, P and K is functional of λ only and R is shock radius.

Substituting the values from equations (20) in equations (5)-(8), we get the following transformed equations

$$U' = \frac{2PUN + \frac{2P}{\Omega}(\delta-1) + \frac{K^2}{\Omega}(2\delta-1) + U(U-1)(\delta-U)}{\lambda\left[(\delta-U)^2 - PN + \frac{K^2}{\Omega}\right]}, \tag{21}$$

$$\Omega' = \frac{\Omega(\lambda U' + 2U)}{\lambda(\delta-U)}, \tag{22}$$

$$P' = \frac{P}{\lambda(\delta-U)}\left[(\lambda U' + 2U)\Omega N + 2(U-1)\right], \tag{23}$$

$$K' = \frac{K\left[\lambda U' - (1-2U)\right]}{\lambda(\delta-U)}, \tag{24}$$

where

$$N = \left[\frac{\alpha}{\{(\gamma-1)+\alpha\Omega\}} + \frac{\gamma}{\Omega} + \alpha\right]; \tag{25}$$

and the prime denotes differentiation with respect to λ.

The transformed shock conditions are

$$U(1) = (1-\beta)\delta, \tag{26}$$

$$\Omega(1) = \frac{1}{\beta}, \tag{27}$$

$$P(1) = (1-\beta)\delta^2, \tag{28}$$

$$K(1) = \frac{1}{M_A}\frac{\delta}{\beta}. \tag{29}$$

RESULTS AND DISCUSSIONS

We have calculated our results for the values of following parameters, $\gamma = 1.4$, $M_A = 10$, $\delta = 0.75$ and $\alpha = 0.0$; 0.025. Numerical results are obtained with the help of Runge-Kutta-Gill method.

We observe that due to non-ideal nature of gas, the nature of flow variables is much affected. We have also compared our results with results of Singh and Shrivastava [5].

From the Table 1 and 2 we can see when $\alpha = 0.0$ the values of velocity, density, pressure and magnetic field are more higher that the values of variables at $\alpha = 0.025$. Velocity in both the cases increases slowly. Density at $\alpha = 0.025$ first decreases and in the last it increases but in case of $\alpha = 0.0$ it always increases. In case of pressure and magnetic field their values always increase.

Table- 1. Alpha $(\alpha) = 0.025$, Beta $(\beta) = 0.2342586$

λ	U	Ω	P	K
1.000	0.5743061	4.2687864	0.4307296	0.3201590
0.990	0.5845472	3.9939592	0.4462961	0.3368078
0.980	0.5954407	3.7305610	0.4647801	0.3564513
0.970	0.6070673	3.4785471	0.4871852	0.3800999
0.960	0.6195424	3.2384248	0.5151604	0.4093501
0.950	0.6330356	3.0117302	0.5516211	0.4469098
0.940	0.6478177	2.8023138	0.6023023	0.4978594
0.930	0.6643681	2.6204002	0.9805022	0.5733038
0.920	0.6837242	2.5002503	0.8273253	0.7048531
0.910	0.7093794	2.6601505	1.2981850	1.0608522
0.904	0.7451237	13.5059175	33.4011002	7.6948133

Table-2. Alpha $(\alpha) = 0.000$, Beta $(\beta) = 0.1166000$

λ	U	Ω	P	K
1.000	0.6625500	8.5763292	0.4969125	0.6432247
0.999	0.6659420	8.6835175	0.5179785	0.6666209
0.998	0.6695619	8.8206387	0.5428787	0.6937802
0.997	0.6734729	8.9995308	0.5730920	0.7260504
0.996	0.6777758	9.2403202	0.6111223	0.7656698
0.995	0.6826475	9.5815.58	0.6617379	0.8168068
0.994	0.6884464	10.1127253	0.7359919	0.888869
0.993	0.6964263	11.1543379	0.8731898	1.0141549

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AN EXACT SIMILARITY SOLUTION FOR THE FLOW BEHIND A STRONG SHOCK WAVE IN A MIXTURE OF A GAS AND DUST PARTICLES

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ABSTRACT

An exact similarity solution for the flow behind a strong shock wave propagating in a mixture of a gas and small solid particles in which the density remains constant and radiation flux is important, has been obtained. The solid particles are considered as a pseudo-fluid and it is assumed that the equilibrium flow condition is maintained. Calculations have been made for different values of the ratio of specific heats of the gas γ, the mass concentration of solid particles k_p, and the ratio of density of solid particles to the initial density of gas G.

Key Words : Shock wave, solid particles, radiation-flux, pseudo fluid.

INTRODUCTION

In many astrophysical and engineering problems, we should consider the high speed flow of a gas and small solid particles [1,2]. We, therefore, generalize the solution of point explosion in gas by Ashraf and Sachdev [3] to the case of two phase flow of a mixture of a gas and small solid particles.

In our study, we have found an exact similarity solution for the flow behind a strong shock (plane, cylindrical or spherical) propagating in a mixture of a gas and small solid particles in which the density remains constant and radiation flux is important. In order to get some essential features of the shock propagation, it is assumed that the viscous stress and heat conduction of the mixture are negligible. Furthermore, the solid particles are considered as a pseudofluid and it is assumed that the equilibrium condition is maintained in the flow-field so that the velocity of the gas, that of the pseudo-fluid of solid particles, and that of the mixture are equal. The temperature of the gas, that of the solid particles, and that of the mixture are also equal [2].

As in [3], the radiation pressure and radiation energy are considered to be very small in comparison to material pressure and energy, respectively, and therefore only

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radiation flux is taken into account. We have made use of the "Product Solutions" of Mc Vittie [3,4] to evaluate the flow variables in the flow-field behind a strong shock moving in a mixture of a gas and small solid particles.

FUNDAMENTAL EQUATIONS

The fundamental equations for the one dimensional and unsteady flow of a mixture of gas and solid particles in which the effect of radiative heat flux may be significant, can be written (c.f. [2], [3]) as

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\frac{\rho}{r^j}\frac{\partial}{\partial r}\left(u.r^j\right)=0 \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0 \tag{2.2}$$

$$\frac{\partial U_m}{\partial t}+u\frac{\partial U_m}{\partial r}+\frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}\right)+\frac{1}{\rho r^j}\frac{\partial}{\partial r}\left(F.r^j\right)=0 \tag{2.3}$$

where

- ρ is the density of mixture,
- u the flow velocity,
- p the pressure,
- U_m the internal energy per unit mass of the mixture,
- F the radiative heat flux per unit mass
- r the radial distance,
- t the time, and $j = 0,1$ or 2

Correspond to the plane, cylindrical or spherical symmetry.

Equation of state of the mixture can be written as

$$p=\frac{1-k_p}{1-Z}(\rho\, R*T) \tag{2.4}$$

where, R^* is the gas constant, Z the volume fraction of solid particles in the mixture and k_p the mass concentration of solid particles. The relation between k_p and Z is [2]

$$k_p = \frac{Z \rho_{sp}}{\rho} \tag{2.5}$$

where $$Z = \frac{Z_1 \rho}{\rho_1}$$

ρ_{sp} is the species density of solid particle and Z_1 and ρ_1 are initial values of Z and ρ respectively. In the equilibrium flow, k_p is constant in the whole flow field.

The internal energy U_m of the mixture may be written as

$$U_m = k_p C_{sp} + (1-k_p) C_v = C_{vm} T \tag{2.6}$$

where C_{sp} is the specific heat of solid particles, C_v the specific heat of the gas at constant volume, and C_{vm} the specific heat of the mixture at constant volume process. The specific heat of the mixture at constant pressure process is

$$C_{pm} = k_p C_{sp} + (1-k_p) C_p \tag{2.7}$$

where C_p is the specific heat of the gas at constant pressure process.

The ratio of specific heats of the mixture is given by

$$\Gamma = \frac{C_{pm}}{C_{vm}} = \gamma \frac{(1+\delta\beta'/\gamma)}{1+\delta\beta'} \tag{2.8}$$

where $$\gamma = \frac{C_p}{C_v}, \delta = \frac{k_p}{1-k_p}, \text{ and } \beta' = \frac{C_{sp}}{C_v}$$

The internal energy of mixture is, therefore, given as [2]

$$U_m = \frac{p}{\rho} \frac{(1-z)}{(\Gamma - 1)} \tag{2.9}$$

We consider that a strong shock wave is propagated into the medium of constant density ρ_1, at rest ($u_1 = 0$) with negligibly small counter pressure ($p_1 = 0$). We assume the shock to be a thin surface so that the radiative heat flux is continuous across it. The boundary

conditions at the strong shock are as follows (c.f. [2] and [5]).

$$u_2 = (1-\beta)\dot{R} \tag{2.10}$$

$$\rho_2 = \frac{\rho_1}{\beta} \tag{2.11}$$

$$p_2 = (1-\beta)\rho_1 \dot{R}^2 \tag{2.12}$$

$$Z_2 = Z_1 / \beta \tag{2.13}$$

where

$$\beta = \frac{\Gamma + 2Z_1 - 1}{\Gamma + 1}$$

Z_1 the initial volume fraction of solid particles, is taken as a constant and R is the shock radius. Dot denotes the differentiation with respect to time t. A quantity with suffix '2' denotes the value of that quantity just behind the shock front. The relation between k_p and Z_1 is given as [5]

$$Z_1 = \frac{k_p}{G(1-k_p)+k_p} \tag{2.14}$$

where G is the ratio of the density of solid particles to the initial density of gas.

The shock radius is assumed to be given as [3]

$$\dot{R}^2 = A^2 R^{-\alpha} \tag{2.15}$$

where A and α are constants.

SIMILARITY SOLUTION

Let the solution of the problem exist in the following similarity form

$$u = \dot{R}\,\bar{u}(x),\ \rho = \rho_1\,\bar{\rho}(x),\ p = \rho_1 R^2\,\bar{p}(x)$$

$$U_m = \dot{R}^2\,\bar{U}_m(x),\ F = \rho\,\dot{R}^3\,\bar{\mathrm{F}}(x) \tag{3.1}$$

where $x = \frac{r}{R}$ is the non-dimensional similarity variable.

The equations of motion (2.1), (2.2) and (2.3) transform into the following form

$$(\bar{u}-x)\frac{\bar{\rho}'}{\bar{\rho}} = -\left(\bar{u}' + j\frac{\bar{u}}{x}\right) \tag{3.2}$$

$$(\bar{u}-x)\,\bar{u}' - \frac{\alpha}{2}\,\bar{u} = -\frac{\bar{p}'}{\bar{\rho}} \tag{3.3}$$

$$(\bar{u}-x)\,\bar{U}_m' - \alpha\bar{U}_m + \frac{\bar{p}}{\bar{\rho}}\left(j\frac{\bar{u}}{x} + \bar{u}'\right) + \frac{1}{\bar{\rho}x^j}\frac{d}{dx}\left(\bar{F}.x^j\right) = 0 \tag{3.4}$$

The strong shock conditions (2.10), (2.11) and (2.12) change into the following form :

$$\bar{u}(1) = (1-\beta) \tag{3.5}$$

$$\bar{p}(1) = (1-\beta) \tag{3.6}$$

$$\bar{\rho}(1) = \frac{1}{\beta} \tag{3.7}$$

We assume the 'Product Solution' of the progressive wave given by Mc Vittie [4] in the form

$$u = \frac{a(t)}{t}\,.r \tag{3.8}$$

$$\rho = (\lambda+1)\,.f.t^{-2\alpha'}\eta^{\lambda-2} \tag{3.9}$$

$$p = \alpha'^2.f.t^{-2}\,b(t)\,.\eta^{\lambda} \tag{3.10}$$

where $\eta = r\,t^{-\alpha'}$, and λ and α' are some constants.

Also, 'a' and 'b' are some functions of t and are given by

$$a(t) = \frac{\alpha'\lambda - tf_t / f}{\lambda + 1} \tag{3.11}$$

$$b(t) = \frac{\lambda + 1}{\lambda\alpha'2}(-a + a - ta_t) \tag{3.12}$$

It can be easily seen that these equations satisfy the equations (2.1) and (2.2) identically.

After changing this solutions to similarity form which requires 'a' to be a constant (equal to $\frac{2(1-\beta)}{\alpha+2}$), we apply boundary conditions (3.5), (3.6) and (3.7) and finally obtain

$$\bar{u}(x) = (1-\beta)\, x \tag{3.13}$$

$$\bar{p}(x) = (1-\beta)\, x^{\lambda} \tag{3.14}$$

$$\bar{\rho}(x) = \frac{1}{\beta} x^{\lambda-2} \tag{3.15}$$

These equations satisfy the differential equations (3.2) and (3.3) identically and therefore, they constitute a solution of these differential equations.

Using equation (2.9), equation (3.4) becomes

$$(\bar{u} - x)\left[\frac{\bar{p}'}{\bar{p}} - \frac{\bar{\rho}'}{\bar{\rho}}\frac{1}{(1-z)}\right] - \alpha + \frac{(\Gamma-1)}{(1-Z)}\left[j\frac{\bar{u}}{x} + \bar{u}'\right] + \frac{(\Gamma-1)}{x^j \bar{p}(1-Z)}\frac{d}{dx}\left(\bar{F}.x^j\right) = 0 \tag{3.16}$$

Using equations (3.13) and (3.15) in (3.2), we obtain

$$\lambda - 1 = \frac{(1-\beta)j + 1}{\beta} \tag{3.17}$$

Using equations (3.13), (3.14), (3.15) and (3.17) in (3.3), we obtain

$$\alpha = 2[(1-\alpha)j + 1] \tag{3.18}$$

From equations (3.13), (3.14), (3.15) and (3.16), we obtain

$$F(x)=\frac{(1-\beta)}{(\Gamma-1)}\left[2\{(\beta+1)+(1-\beta)j\}-(\Gamma-1)(1-\beta)(1+j)\right]\frac{x^{\lambda+1}}{\lambda+j+1}$$

$$-\frac{(1-\beta)Z_1}{\beta(\Gamma-1)}\left[\beta\lambda+2+2(1-\beta)j\right]\frac{x^{2\lambda-1}}{2\lambda+j-1} \qquad (3.19)$$

where $\lambda=\dfrac{1+(1-\beta)j+}{\beta}$.

We also have the relations $\dfrac{u}{u_2}=\dfrac{\bar{u}(x)}{\bar{u}(1)}$, $\dfrac{p}{p_2}=\dfrac{\bar{p}(x)}{\bar{p}(1)}$, $\dfrac{\rho}{\rho_2}=\dfrac{\bar{\rho}(x)}{\bar{\rho}(1)}$, $\dfrac{F}{F_2}=\dfrac{\bar{F}(x)}{\bar{F}(1)}$,

$Z=\dfrac{Z_1}{\beta}\left(\dfrac{\rho}{\rho_2}\right)$ and $\dfrac{T}{T_2}=\dfrac{p}{p_2}\dfrac{(1-Z)}{(1-Z_2)}\Big/\left(\dfrac{\rho}{\rho_2}\right)$.

Equations (3.13), (3.14), (3.15) and (3.19) give the solution of our problem. This solution is an example of exact solution for the flows of mixture of a gas and small solid particles corresponding to exact solution in ordinary gas dynamics by Mc Vittie [4] and Se(ɔv [6], in radiation gas dynamics by Ashraf and Sachdev [3],in magnetogasdynamics by Ojha, Nath and Takhar [7], and for flows of water by Vishwakarma and Mishra [8].

RESULTS AND DISCUSSION

For the density to be remain finite at the centre and for the flux not to be negative anywhere, we have, from (3.13), (3.14), (3.15) and (3.19),

$$\beta+1+(1-\beta)j>2\beta$$

and $2\{\beta+1+(1-\beta)j\}-(\Gamma-1)(1-\beta)(1+j)-\dfrac{\beta\lambda+2+(1-\beta)j}{\beta}>0.$

In Figures 1 to 4 and Tables 1 to 2, we have tabulated the values of $\dfrac{\rho}{\rho_2}$, $\dfrac{p}{p_2}$, $\dfrac{F}{F_2}$, $\dfrac{T}{T_2}$

and z for $j=1$; $y=\dfrac{7}{5}$, $\dfrac{5}{3}$; $K_p=0.1,\ 0.4,\ G=1,\ 100,$ [2], and $\beta'=1$ [9]).

This solution predicts velocity, density, pressure and radiation flux to be zero at the axis of summetry. The values of all physical quantities decrease from highest at the shock to zero at the axis.

Since $\frac{u}{u_2} = x$, it does not vary with any variation in γ, k_p or G. In figures 1, 2, we find that for a given k_p and G the flow variables $\frac{\rho}{\rho_2}$ and $\frac{p}{p_2}$ increase as γ increases. From figure 3, we find that the flux $\frac{F}{F_2}$ increases with increase of γ, except for the case of larger k_p, smaller G ($k_p = 0.4$, $G = 1$) and near the shock,where it decreases with increase of γ. We find from tables 1 & 2 that the volume fraction of solid particles Z decreases near the shock front with increase of γ. Also, from figure 4 temperature decreases $\frac{T}{T_2}$ decreases with increase of γ. From figures 1 to 4, and tables 1 and 2, we see that, for given γ and smaller G (G=1), $\frac{\rho}{\rho_2}$, $\frac{p}{p_2}$, $\frac{F}{F_2}$, Z, $\frac{T}{T_2}$ increase with increase of k_p. Also, we find that for larger G (G=100), $\frac{p}{p_2}$, $\frac{\rho}{\rho_2}$, $\frac{F}{F_2}$ decreases with increase of k_p, while z decreases near the axis but increase near the shock front with increase of k_p. Also $\frac{T}{T_2}$ increases for larger G and smaller γ $(G = 100,\ \gamma = 1.4)$ and decreases for larger G and large $\gamma \left(G = 100,\ \gamma = \frac{5}{3}\right)$ with increase of K_p. Further, we find that for given γ and k_p, $\frac{\rho}{\rho_2}$, $\frac{p}{p_2}$, $\frac{F}{F_2}$ and Z decrease with increase of G. $\frac{T}{T_2}$ also decreases with increase of G, except for the case of smaller k_p and larger $\gamma \left(k_p = 0.1,\ \gamma = \frac{5}{3}\right)$ where it increases with increase of G.

Thus our exact solution shows that the shock propagation in two phase flow of a mixture of a gas and small solid particles is much more complicated than that in ordinary gas dynamics.

Table 1. Variation of Z with x for j=1; G=1; γ=1; $\gamma = \frac{7}{5}, \frac{5}{3}; k_p = 0.1, 0.4$

x	Z			
	$\gamma = \frac{7}{5}, k_p = 0.1$	$\gamma = \frac{7}{5}, k_p = 0.4$	$\gamma = \frac{5}{3}, k_p = 0.1$	$\gamma = \frac{5}{3}, k_p = 0.4$
0.1		0.00420	0.00001	
0.3	0.00018	0.00534	0.00144	
0.5	0.00489	0.17372	0.01436	0.00002
0.7	0.04253	0.37796	0.06528	0.00124
0.9	0.21403	0.66754	0.20228	0.01392
1.0	0.42141	0.86152	0.32499	0.03832

Table 2. Variation of Z with x for j=1; G=100; $\gamma = \frac{7}{5}, \frac{5}{3}; k_p = 0.1, 0.4$

x	Z			
	$\gamma = \frac{7}{5}, k_p = 0.1$	$\gamma = \frac{7}{5}, k_p = 0.4$	$\gamma = \frac{5}{3}, k_p = 0.1$	$\gamma = \frac{5}{3}, k_p = 0.4$
0.5			0.00004	0.00002
0.7	0.00014	0.00215	0.00044	0.00124
0.9	0.00224	0.01120	0.00235	0.01392
1.0	0.00716	0.05840	0.00474	0.03832

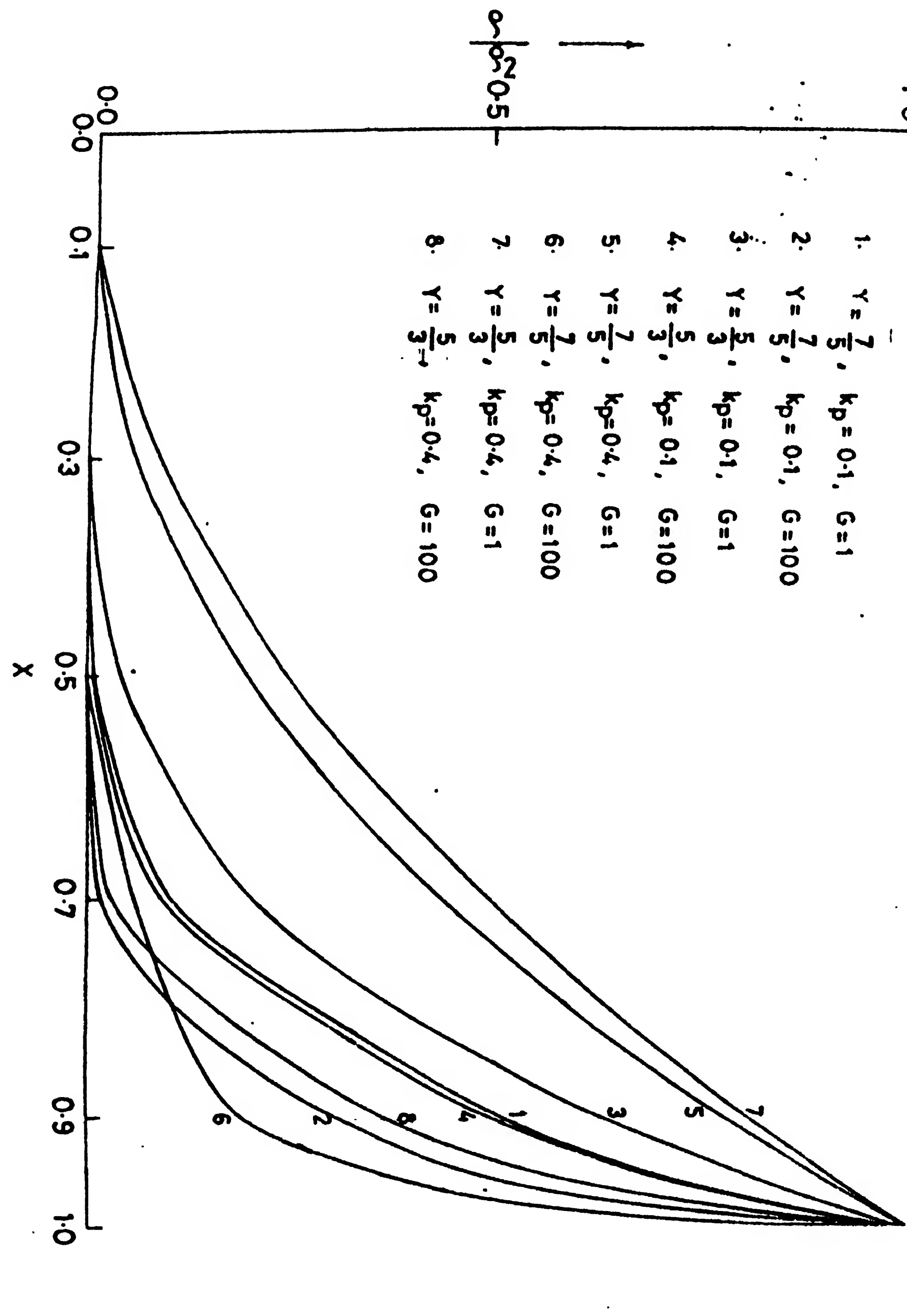

Fig. 1 Variation of density with distance

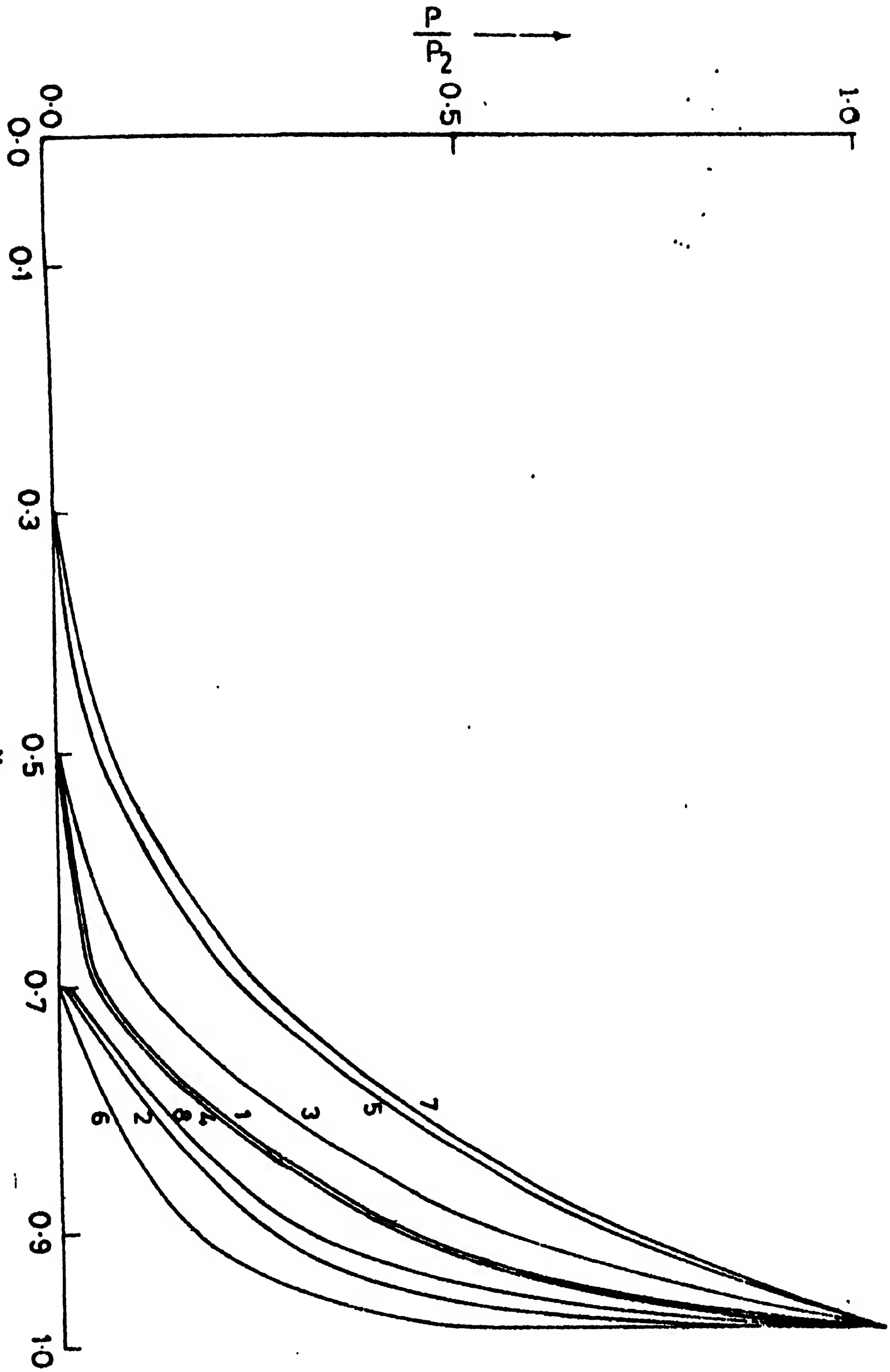

Fig. 2 Variation of pressure with distance

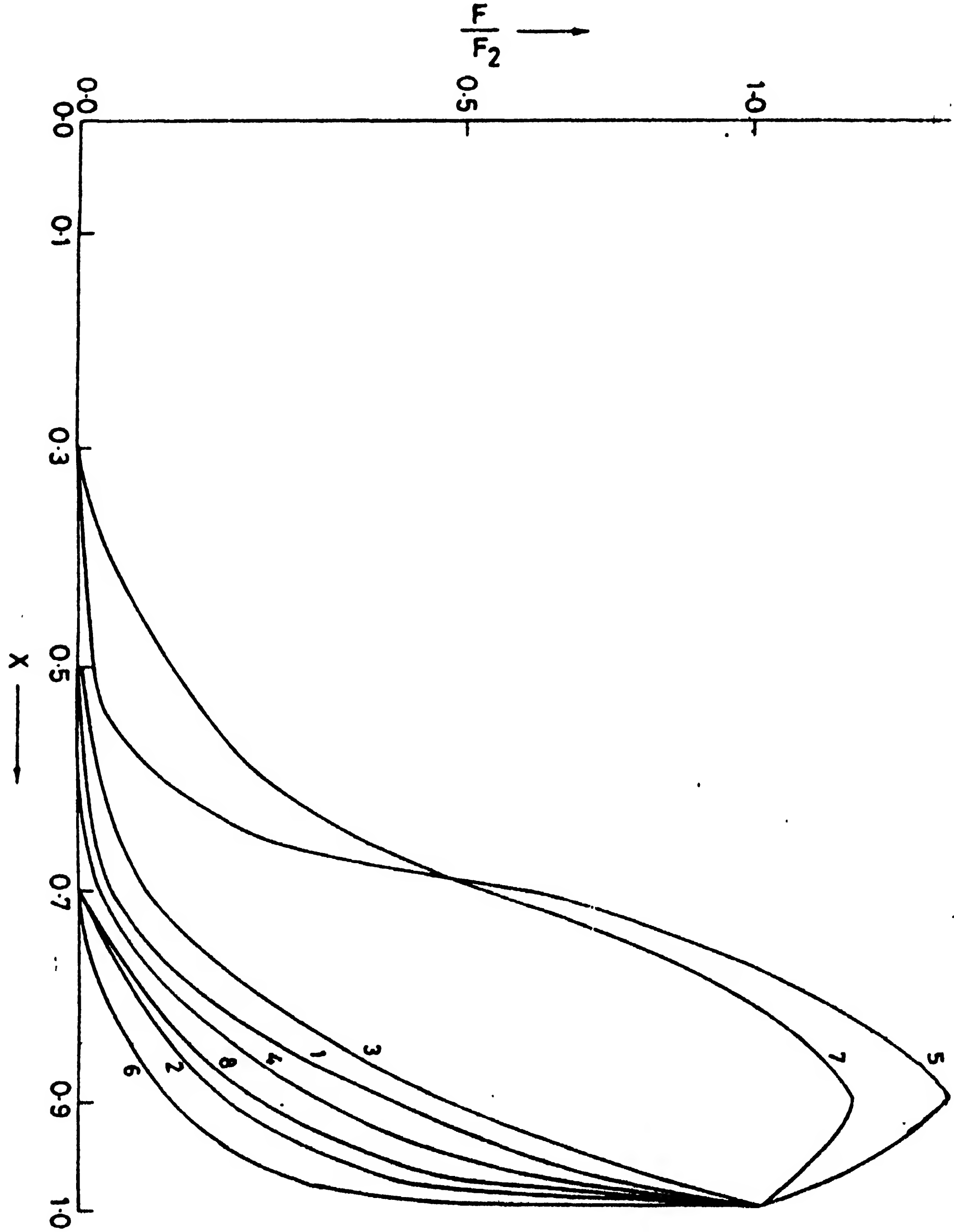

Fig. 3 Variation of radiation flux with distance

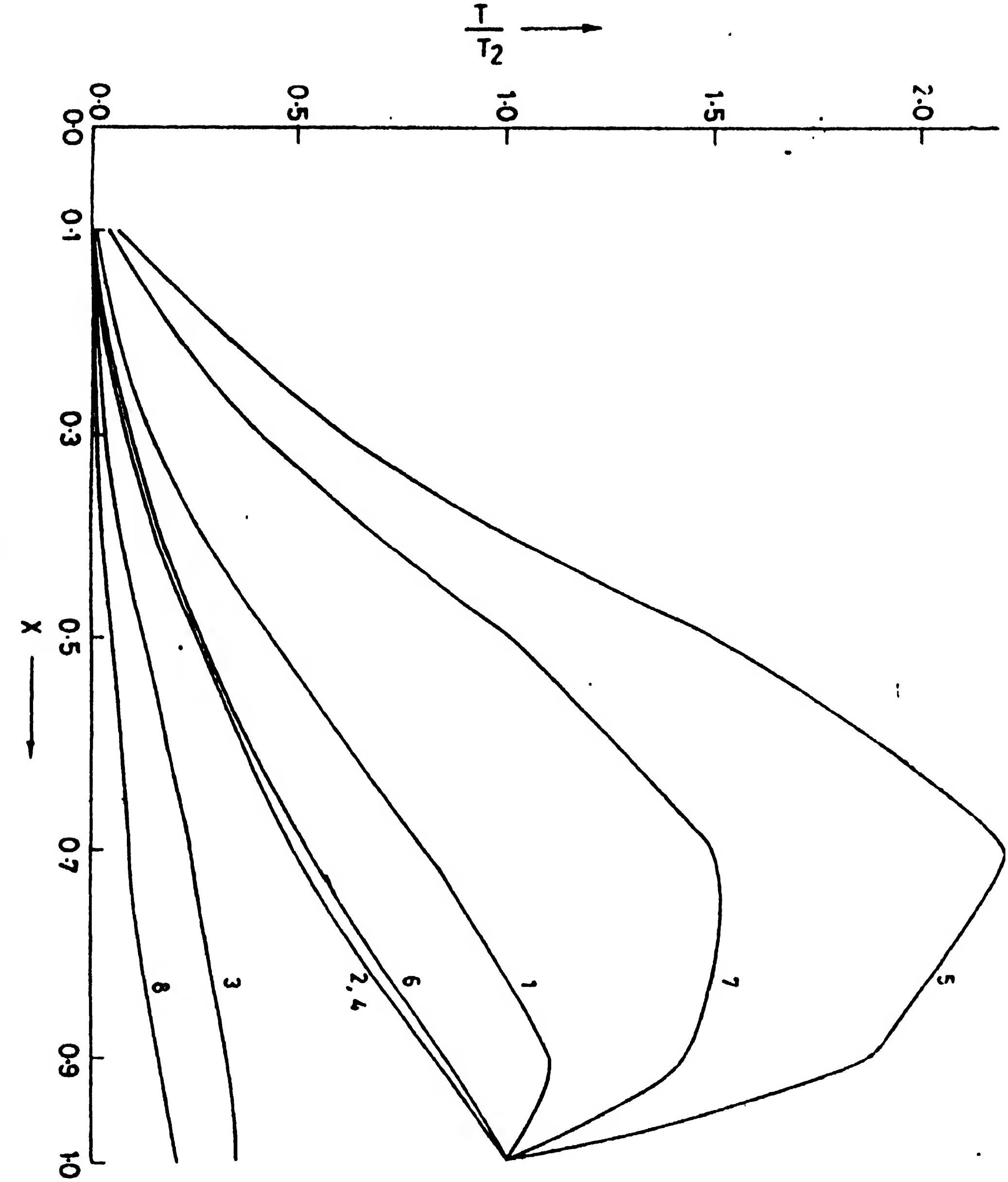

Fig. 4 Variation of temperature with distance

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ON QUASI-CONTRACTION MAPPINGS IN D-METRIC SPACES

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ABSTRACT

A class of pairs of generalized contraction mappings on a D-metric space is introduced and studied for their common fixed points. A common fixed points theorem for pairs of coincidentally commuting mappings on a D-metric space satisfying certain generalized contraction condition is proved. Our results include the fixed point theorems of Dhage [5] as spacial cases.

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Key Words & Pharases : D-metric spaces, coincidentally commuting maps fixed point, etc.

INTRODUCTION

Motivated by the measures of nearness, recently the present author [1] introduced the notion of a D-metric space as follows. Let X denote a non empty set and R^+, the set of all non-negative real numbers. Then X together with a function $\rho: X \times X \times X \to \mathbf{R}^+$ is called a D-metric space and denoted by the pair (X, ρ) if ρ satisfies the following properties :

(i) $\rho(x,y,z) = 0 \Leftrightarrow x = y = z$, (coincidence)

(ii) $\rho(x,y,z) = \rho(p\{x,y,z\})$, (symmetry)

p denotes the permutation function, and

(iii) $\rho(x,y,z) \le \rho(x,y,a) + \rho(x,a,z) + \rho(a,y,z)$

for $x,y,z,a \in X$. (tetrahedral inequality)

A sequence $\{x_n\} \subset X$ is said to be *convergent* and converges to a point x if $\lim_{m,n} \rho(x_m, x_n, x) = 0$. A sequence $\{x_n\} \subset X$ is called *D-Cauchy* if $\lim_{m,n,p} \rho(x_m, x_n, x_p) = 0$. A complete D-metric space is one in which every D-Cauchy

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sequence convergent to a point in it. A set $S \subset X$ is said to be bounded if their exists a constant $M > 0$ such that $\rho(x,y,z) \le M$ for all $x,y,z \in S$ and the constant M is called a *D-bound* of S. It is known that the D-metric ρ is a continuous function on X^3 in the topology of D-metric convergence which is Hausdoff in nature. Some details along with some specific examples of the D-metric space appear in [3].

The Banach contraction mapping principle analogue proved by Dhage [1] in a D-metric space is as follows.

Theorem 1.1 : Let X be a complete and bounded D-metric space and let $f: X \to X$ satisfy

$$\rho(fx, fy, fz) \le \alpha\rho(x,y,z) \tag{1.1}$$

for all $x,y,z \in X$ and $0 \le \alpha < 1$. Then f has a unique fixed point.

Let $f: X \to X$. Then an orbit of the function f at a point $x \in X$, is a set $O(x)$ in X given by

$$O(x) = \{x, fx, f^2x, \ldots\}. \tag{1.2}$$

By $\overline{O(x)}$, we mean the closure of the orbit $O(x)$, $x \in X$ in X, A D-metric space X is said to be f-orbitally bounded if the orbit $O(x)$ is bounded for each $x \in X$. A D-metric space X is said to be f-orbitally complate if every D-Cauchy sequence $\{x_n\} \subseteq O(x)$, $x \in$ X converges to a point in X. A mapping $f: X \to X$ is called f-orbitally continuous if $\{x_n\} \subseteq O(x)$, $x \in X$, $x_n \to x^*$ implies $fx_n \to fx^*$.

The conclusion of Theorem 1.1 under some weaker condition obtained recently by Dhage and Rhoades [7] is as follows.

Theorem 1.2 : Let f be a self-map of f-orbitally complete D-metric space X satisfying (1.1). Then f has a unique fixed point x^* and the sequence $\{f^n x\}$, $x \in X$, of successive iteration converges to x^*.

Recently two more basic contraction mapping principles have been obtained by the present auother [5] for the self mappings f and g of D-metric space X characterized by the inequalities

$$\rho(fx, fy, z) \leq \alpha\rho(x, y, z) \tag{1.3}$$

for all $x, y \in X$ and $z \in \overline{O(x) \cup O(y)}$, where $0 \leq \alpha < 1$.

and $$\rho(fx, fy, z) \leq \alpha\rho(gx, gy, z) \tag{1.4}$$

for all $x, y \in X$, with some restriction on z, and $0 \leq \alpha < 1$.

Obviously condition (1.3) does not necessarily imply condition (1.1) on a D-metric space X.

To see this, let $z \in \overline{O(x) \cup O(y)} \cap \overline{O(y) \cup O(w)} \cap \overline{O(w) \cup O(x)}$, then by condition (1.3)

$$\rho(fx, fy, fw) \leq \rho(fx, fy, z) + \rho(fx, fw, z) + \rho(fy, fw, z)$$

$$\leq \alpha\rho(x, y, z) + \alpha\rho(x, w, z) + \alpha\rho(y, w, z)$$

$$= \alpha[\rho(x, y, z) + \rho(x, z, w) + \rho(z, w, y)\ ]$$

$$\not\leq \alpha\,\rho(x, y, w)$$

and so condition (1.1) is not satisfied.

In this paper we shall prove some fixed point theorems for a class of single as well as pairs of mappings on a D-metric space which is wider than given by (1.3) and (1.4) there by generalizing the results of Dhage [5].

PRELIMINARIES

Before going to main results, we give some preliminaries needed in the sequel.

Lemma 2.1 (***D-Cauchy Principle***): Let $\{x_n\} \subset X$ be a bounded sequence with D-bound k satisfing

$$\rho(x_n, x_{n+1}, x_m) \leq \phi^n(k) \tag{2.1}$$

for all $m > n \in N$, where $\phi : R^+ \to R^+$ satisfies $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each t $\in \mathbf{R}^+$ then $\{x_n\}$ is D-Cauchy.

***Proof*:** Since $\sum_{n=1} \phi^{n}(k)$ is a convegent sequence of nonnegative real numbers, we have

$$\lim_{n} \phi^{n}(k) = 0 \text{ and } \lim_{m,n} \sum_{j=n+1}^{m} \phi(k) = 0$$

Now for any $p,\ t \in N$, by (2.1) we get

$$\left.\begin{array}{l} \rho(x_{n,} x_{n+1,} x_{n+p}) \leq \phi^{n}(k) \\ \rho(x_{n,} x_{n+1,} x_{n+p+t}) \leq \phi^{n}(k). \end{array}\right\}$$

Then by repeated application of the tetrahedral inequality we get

$$\begin{aligned}
&\rho(x_{n,} x_{n+1,} x_{n+p+t}) \\
&\leq \rho(x_{n,} x_{n+1,} x_{n+p+t}) + \rho(x_{n,} x_{n+p,} x_{n+1}) + \rho(x_{n+1,} x_{n+p,} x_{n+p+t}) \\
&\leq 2\phi^{n}(k) + \rho(x_{n+1,} x_{n+p,} x_{n+p+t}) \\
&\leq 2\phi^{n}(k) + \rho(x_{n+1,} x_{n+2,} x_{n+p+t}) + \rho(x_{n+1,} x_{n+p,} x_{n+2}) \\
&\qquad + \rho(x_{n+2,} x_{n+2,} x_{n+p+t}) \\
&\leq 2[\phi^{n}(k) + \phi^{n+1}(k)] + \rho(x_{n+2,} x_{n+p,} x_{n+p+t}) \\
&\leq \cdots \\
&\leq 2 \sum_{j=n}^{n+p-1} \phi^{j}(k) + \rho(x_{n+p-1,} x_{n+p,} x_{n+p+t}) \\
&\leq 2 \sum_{j=n}^{n+p-1} \phi^{j}(k) + 2 \sum_{j=n+1}^{n+p+t-1} \phi^{j}(k) + \rho(x_{n+p-1,} x_{n+p,} x_{n+p+t}) \\
&\leq 2 \sum_{j=n}^{n+p+t} \phi^{j}(k) + \phi^{n+p-1}(k) \\
&\to 0 \ as\ n \to \infty.
\end{aligned}$$

This shows that $\{y_n\}$ is D-Cauchy and the proof of the lemma is complete.

Let Φ denote the class of all functions $\phi : R^+ \to R^+$ satisfying

(i) ϕ is continuous,

(ii) ϕ is nondecreasing,

(iii) $\phi(t) < t$ for each $t > 0$, and

(iv) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in R^+$

There do exist functions ϕ satisfying the conditions (i) - (iv) mentioned above. For example, the function $\phi : R^+ \to R^+$ defined by $\phi(t) = \alpha t$, $0 \leq \alpha < 1$ belongs to Φ.

Lemma 2.2: if $\phi \in \Phi$, then $\phi^n(0) = 0$ for each $n \in N$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$.

***Proof*:** if $\phi \in \Phi$, then form (iv) it follows that $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$. Suppose that $\phi(0) \neq 0$. Then $0 < \phi(0) = t$ for some $t > 0$. Since ϕ is nondecreasing, we have

$$0 < \phi(0) = t \leq \phi(t) \leq \phi^2(t) \leq \dots \leq \phi^n(t).$$

Passing the limit to $n \to \infty$, we obtain $0 < \lim \phi^n(t) = 0$, a contradiction. Hence $\phi^n(0) = 0$ for each $n \in N$ and the proof of the lemma is complete.

COMMUTING MAPS AND FIXED POINT PRINCIPLES

In this section we prove some fixed point principles concerning the common fixed points for pairs of mappings in D-metric spaces.

Two mappings $f, g : X \to X$ are said to be <u>**commuting**</u> if $(fg)(x) = (gf)(x)$ for all $x \in X$ and <u>**coincidentally commuting**</u> if they commute at coincidence points. It is clear that every commuting pair is coincidentally commuting, but the following simple example shows that the converse may not be true.

Example 3.1: Define two mappings $f, g : [0,1] \to [0,1]$ by $f(x) = x/2 + 1/2$ and $g(x) = x^2$, $x \in [0,1]$. Obviously $f(g(x)) \neq g(f(x))$ for all $x \in [0,1]$, because $f(g(0)) = 1/2 \neq 1/4 = g(f(0))$. But the only common coincidence point of f and g in [0,1] is 1, and $f(g(1)) = g(f(1))$. This shows that f and g are coincidentally commuting, but not

commuting on R^+.

Let $f: X \to g(X)$, $g: X \to X$ and define and (f/g)-orbit of f and g at a point $x \in X$ to be the set $O(f/g,x)$ in X defined by

$$O(f/g,x) = \{x\} \cup \{y_n / x_0 = x, y_0 = gx_0, y_n = fx_{n-1} = gx_n, n \in \mathrm{N}\}$$

for some sequence $\{x_n\}$ in X.

Denote by $\overline{O(f/g,x)}, x \in X$, the closure of $O(f/g, x)$ *in* X.

Theorem 3.1 Let $f, g: X \to X$, be two mappings satisfying

$$\rho(fx, fy, z) \leq \phi(\max\{\rho(gx, gy, z), \rho(gx, fx, z), \rho(gy, fy, z), \rho(gx, fy, z)\}) \tag{3.1}$$

for all $x, y \in X$ *and* $z \in \overline{O(f/g,x) \cup O(f/g,y)}$; *where* $\phi \in \Phi$. Further suppose that

(i) $f(X) \subseteq g(X)$,

(ii) $f(X)$ is bounded and $g(X)$ is complete, and

(iii) f and g are coincidentally commuting.

Then f and g have a unique common fixed point u, and if g is continuous at u, then f is also continuous at u.

Proof: Let $x \in X$ be arbitrary and define a sequence $\{y_n\} \in X$ *by*

$$y_0 = gx_0, / x_0 = x, / y_{n+1} = fx_n = gx_{n+1}, n = 0, 1, 2, ... \tag{3.2}$$

Clearly the sequence $\{y_n\}$ is well defined since $f(X) \subseteq g(X)$. Moreover $\{y_n\} \subset g(X)$.

Now there are two cases :

Case I: If $y_r = y_{r+1}$ for some $r \in N$, then we have

$$fx_{r-1} = fx_r = gx_r = gx_{r+1}. \tag{3.3}$$

We shall now show that $fx_r = gx_{r+1} = \mathrm{u}$ is a common fixed point of f and g. By virtue of the coincidentally commutativity of f and g, from (3.1) it follows that $fu = gu$.

Now $\rho(fu, gu, u)$

$$= \rho(fu, fgx_r, u)$$

$$\leq \phi\,(max\,\{\rho(gu, ggx_r, u), \rho(gu, fu, u), \rho(ggx_r, fgx_r, u), \rho(gu, fgx_r, u), \rho(ggx_r, fu, u)\})$$

$$= \phi\,(max\,\{\rho(gu, gu, u), \rho(gu, fu, u), \rho(gu, fu, u), \rho(gu, fu, u), \rho(gu, fu.u)\}),$$

$$= \phi\,(\rho(gu, gu, u))$$

which implies that $fu = gu = u$ since $\phi \in \Phi$,

Thus f and g have a common fixed point.

__Case II__: Suppose that $y_n \neq y_{n+1}$ for each $n \in N$. Let $x = x_0$, $y = x_1$ and $z = y_m$ $O(f/g, x)$, then by (3.1) we obtain

$$\rho\,(y_1, y_2, y_m)$$

$$= \rho(fx_0, fx_1, y_m)$$

$$\leq \phi\,(max\{\rho(gx_0, gx_1, y_m), \rho(gx_0, fx_0, y_m), \rho(gx_1, fx_1, y_m), \rho(gx_0, fx_1, y_m), \rho(gx_1, fx_0, y_m)\})$$

$$= \phi\,(max\{\rho(y_0, y_1, y_m), \rho(y_0, y_1, y_m), \rho(y_1, y_2, y_m), \rho(y_0, y_2, y_m), \rho(y_1, y_1, y_m)\})$$

$$= \phi\,(\max_{\substack{0 \leq a \leq 1 \\ 0 \leq b \leq 2}} (y_a, y_b, y_m))$$

$$\leq \phi\,(k).$$

Again for $m > 2$,

$$\rho\,(y_2, y_3, y_m)$$

$$= \rho(fx_1, fx_2, y_m)$$

$$\leq \phi\,(max\{\rho(gx_1, gx_2, y_m), \rho(gx_1, fx_1, y_m), \rho(gx_2, fx_2, y_m), \rho(gx_1, fx_2, y_m), \rho(gx_2, fx_1, y_m)\})$$

$$= \phi\,(max\{\rho(y_1, y_2, y_m), \rho(y_1, y_2, y_m), \rho(y_2, y_3, y_m),$$
$$\rho(y_1, y_3, y_m), \rho(y_2, y_2, y_m)\})$$
$$= \phi\,(max\,\{\rho(y_1, y_2, y_m), \rho(y_1, y_3, y_m), \rho(y_2, y_2, y_m)\}) \tag{3.4}$$

But

$$\rho\,(y_2, y_3, y_m) = \rho(fx_1, fx_2, y_m)$$
$$\leq \phi\,(max\{\rho(y_0, y_1, y_m), \rho(y_0, y_1, y_m), \rho(y_1, y_2, y_m),$$
$$\rho(y_0, y_2, y_m), \rho(y_2, y_2, y_m)\})$$

$$= \phi\left(\max_{\substack{0\leq a\leq 1\\ 1\leq b\leq 2}} \rho(y_a, y_b, y_m)\right) \tag{3.5}$$

$$\rho\,(y_1, y_3, y_m) = \rho(fx_0, fx_2, y_m)$$
$$\leq \phi\,(max\{\rho(y_0, y_2, y_m), \rho(y_0, y_1, y_m), \rho(y_2, y_3, y_m),$$
$$\rho(y_0, y_3, y_m), \rho(y_2, y_1, y_m)\})$$
$$\leq \phi\,(\max_{\substack{0\leq a\leq 1\\ 0\leq b\leq 2}} \rho\,(y_a, y_b, y_m)) \tag{3.6}$$

and $\rho\,(y_2, y_2, y_m)$

$$= \rho(fx_1, fx_1, y_m)$$
$$\leq \phi\quad(max\{\rho(y_1, y_1, y_m), \rho(y_1, y_2, y_m), \rho(y_1, y_2, y_m),$$
$$\rho(y_1, y_2, y_m)\})$$
$$= \phi\quad(\max_{\substack{a=1\\ 1\leq b\leq 2}} \rho\,(y_a, y_b, y_m))$$

$$\leq \phi\quad(\max_{\substack{0\leq a\leq 1\\ 1\leq b\leq 2}} \rho\,(y_a, y_b, y_m)) \tag{3.7}$$

Substituting these estimates in (3.4) we get

$$\rho\,(y_2, y_3, y_m) \leq \phi\,(\phi\,(\max_{\substack{0\leq a\leq 2\\ 0\leq b\leq 3}} \rho\,(y_a, y_b, y_m)))$$

$\leq \phi^2 (k).$

Thus in general

$$\rho (y_n, y_{n+1}, y_m) \leq \phi^n (\max_{\substack{0 \leq a \leq 2 \\ 0 \leq b \leq 3}} \rho (y_a, y_b, y_m))$$

$$\leq \phi^n (k) \ (\because f(X) \text{ is bounded})$$

for all $m > n \in N$.

Now an application of Lemma 2.1 yields that $\{y_n\}$ is D-Cauchy. Since $g(x)$ is complete, there is a point $u \in g(X)$ such that $\lim_n y_n = u$,

$$i.e. \lim_n fx_n = \lim_n fx_{n+1} = u = \lim_n gx_{n+1}. \tag{3.8}$$

By the definition of u, there is a point $w \in X$ such that $gw = u$. We show that $fw = gw$.

Now

$$\rho (fw, gw, gw)$$

$$= \lim_n \rho(fw, fx_n, gw)$$

$$\leq \lim_n \phi (max\{\rho(gw, gx_n, gw), \rho(gw, fw, gw), \rho(gx_n, fx_n, gw), \rho(gw, fx_n, gw), \rho(gx_n, fw, gw)\})$$

$$= \phi (max\{0, \rho(fw, gw, gw), 0, 0, 0\})$$

$$= \phi (\rho(fw, gw, gw))$$

which implies that $fw = gw$ since $\phi \in \Phi$.

We show that $fw = gw = u$ is a common fixed point of f and g.

Now by the coincidentally commutativity of f and g, we obtain $fgw = gfw$, *i.e.*, $fu = gu$. From (3.1),

$$\rho (fu, gu, u)$$

$$= \rho(fu, fu, u)$$

$$\leq \phi(\max\{\rho(gu, gu, u), \rho(gu, fu, u), \rho(gu, fu, u), \rho(gu, fu, u), \rho(gu, fu, u)\})$$

$$= \phi(\max\{\rho(gu, gu, u), \rho(gu, fu, u)\})$$

$$= \phi(\rho(fu, gu, u))$$

Which implies that $fu = gu = u$.

Thus in both the cases f and g have a common fixed point.

To prove uniqueness, let v ($\neq u$) be another common fixed point of f and g. Then by (3.1),

$$\rho(u, v\, u) = \rho(fu, fv, u)$$

$$\leq \phi(\max\{\rho(gu, gv, u), \rho(gu, fu, u), \rho(gv, gv, u), \rho(gu, fv, u), \rho(gv, fu, u)\})$$

$$= \phi(\max\{\rho(v, u, v), \rho(v, v, u)\}).$$

If the maximum is $\rho(u, v, u)$, then one obtains

$$\rho(u, v, u) \leq \phi(\rho(u, v, u)) < \rho(u, v, u),$$

a contradiction. Therefore

$$\rho(u, v, u) \leq \phi(\rho(v, v, u)) \tag{*}$$

Interchanging the roles of u and v in the above arguments yields the inequality

$$\rho(u, v, u) \leq \phi(\rho(u, v, u)) \tag{**}$$

Substituting (**) in (*) gives

$$\rho(u, v, u) \leq \phi^2(\rho(u, v, u))$$

a contradiction.

Next we shall show that f is continuous at u. Let $\{z_n) \subset X$ be any sequence such that $z_n \to u$. To finish, it is enough to prove that $fz_n \to fu$, i.e. $\lim_{m,n} \rho(fz_m, fz_n, fu) = 0$. From the continuity of g at u, it follows that $gz_n \to gu$.

Now

$$\rho(fz_m, fz_n, fu) = \rho(fz_m, fz_n, u)$$
$$\leq \phi\,(max\{\rho(gz_m, gz_n, u), \rho(gz_m, fz_m, u), \rho(gz_n, fz_n, u),$$
$$\rho(gz_m, fz_n, u), \rho(gz_n, fz_m, u)\}).$$

Therefore

$$\lim_{m,n} \sup \rho(fz_m, fz_n, fu)$$
$$\leq \phi\,(max \lim_{m} \sup \rho(gu, fz_m, u), \lim_{n} \sup \rho\,(gu, fz_n, u)\}). \tag{3.9}$$

But

$$\lim_{n} \sup \rho\,(gu, fz_n, u) = \lim_{n} \sup \rho\,(fu, fz_n, u)$$
$$\leq \lim_{n} \sup \phi\,(max\{\rho(gu, gz_n, u), \rho(gu, fu, u),$$
$$\phi(\max\,\{0, 0 \lim_{n} \sup \rho\,(gu, fz_n, u)\})$$
$$= \phi\,(\lim_{n} \sup \rho\,(gu, fz_n, u)\,)$$

which implies that $\lim_{n} \sup \rho\,(gu, fz_n, u) = 0$. Similarly

$$\lim_{n} \phi\; \rho\,(gu, fz_n, u) = 0$$

Substituting these estimates in (3.9) we obtain

$$\lim_{m,n} \phi\; \rho(fz_m, fz_n, fu) \leq \phi\,(0) = 0.$$

Since $\lim_{m,n} inf\, \rho(fz_m, fz_n, fu) \leq \lim_{m,n} \sup \rho(fz_m, fz_n, fu)$,

one obtains

$$\lim_{m,n} \rho(fz_m, fz_n, fu) = 0$$

This completes the proof.

Remark 3.1: In [6] the present author proved a common fixed theorem for pairs of mappings $f, g: X \to X$ satisfying the contraction condition of the type

$$\rho(fx, fy, fz) \le \phi(\max\{\rho(gx, gy, gz), \rho(gx, fx, gz), \rho(gy, fy, gz),$$
$$\rho(gx, fy, gz), \rho(gy, fx, gz)\}) \qquad \text{(D)}$$

for all $x, y, z \in X$, where $\phi \in \Phi$. Obviously condition (3.1) does not imply condition (D). Then in view of Example 1.1, the class of pair of contraction mappings f and g given by the inequality (3.1) is independent of the class of mappings f and g characterized by the inequality (D) on a complete and bounded D-metric space X.

Corollary 3.1: Let f and g be two self-maps of a metric space X, p and m are positive integers, satisfying

$$\rho(f^p x, f^p y, z) \le \phi(\max\{\rho(g^m x, g^m y, z), \rho(g^m x, f^p x, z), \rho(g^m y, f^p y, z),$$
$$\rho(g^m x, f^p y, z), \rho(g^m y, f^p x, z)\}) \qquad (3.10)$$

for all $x, y \in X$ and $z \in \overline{O(f^m / g^m, x) \cup O(f^m / g^m, y)}$, where $\phi \in \Phi$.

Further suppose that

(i) $f^p(X) \subseteq g^m(X)$

(ii) $f^p(X)$ is bounded and $g^m(X)$ is complete, and

(iii) f and g are commuting.

Then f and g have a unique common fixed point u. Further if g^m is continuous at u, then f^p is also continuous at u.

Proof : By Theorem 3.1, f^p and g^m have a unique common fixed point, u(say). Then by the commutativity of f and g, we get

$$f^p(fu) = f(f^p u) = fu \text{ and } g^m(fu) = f(g^m u) = fu.$$

This shows that fu is again fixed point of f^p and g^m. By the uniqueness of u, we get $fu = u$. Similarly it is proved that $gu = u$. Again the continuity of f^p at u follows from condition (3.10). The proof is complete.

Corollary 3.2: Let $f: X \to X$, X an f-orbitally complete and f-orbitally bounded D-metric space, satisfying

$$\rho(fx, fy, z) \leq \phi(\max\{\rho(x, y, z), \rho(x, fx, z), \rho(y, fy, z), \rho(x, fy, z), \rho(y, fx, z)\}) \tag{3.11}$$

for all $x, y \in X$ and $z \in \overline{O(x) \cup O(y)}$, where $f \in F$. Then f has a unique fixed point u and f is continuous at u.

Corollary 3.3 : Let $f: X \to X$, X an f-orbitally complete and f-orbitally bounded D-metric space, satisfying

$$\rho(fx, fy, z) \leq \alpha\, \rho(x, y, z) \tag{3.12}$$

for all $x, y \in X$ and $z \in \overline{O(x) \cup O(y)}$, where $0 \leq \alpha < 1$.

Then f has a unique fixed point u and f is continuous at u.

Corollary 3.4 : Let $f: X \to X$, X an f-orbitally bounded and f-orbitally complete D-metric space, p a positive integer, satisfying

$$\begin{aligned}\rho(f^p x, f^p y, z) \leq \phi(\max\{&\rho(x, y, z), \rho(x, f^p x, z), \rho(y, f^p y, z), \\ &\rho(x, f^p y, z), \rho(y, f^p x, z)\})\end{aligned} \tag{3.13}$$

for all $x, y \in X$ and $z \in \overline{O(x) \cup O(y)}$, where $\phi \in \Phi$. Then f has a unique fixed point u, f^p is continuous at u and f is f-orbitally continuous at u.

Below we state some fixed point results for a class of mappings wider than given by (3.11) via the measure of noncompactness. The measure of noncompactness of a bounded set A in X is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf\left\{r > 0 : A = \bigcup_{i=1}^{n} A_i,\ \mathrm{diam}\,(A_i) \leq r \text{ for all } i\right\} \tag{3.14}$$

where diam $(A_i) = \sup\{\rho(x, y, z) \mid x, y, z \in A_i\}$.

Then the measure α of noncompactness enjoys the following properties :

($\alpha 1$) $\alpha(A) = 0 \Leftrightarrow A$ is precompact,

($\alpha 2$) $\alpha(A) = \alpha(\overline{A})$, $\overline{A}$ denotes the closure of A,

($\alpha 3$) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$ and

$(\alpha 4)$ $\alpha(\{A \cup B\}) = \max\ \{\ \alpha(A), \alpha(B)\}$

Definition 3.1: A mapping $f: X \to X$ is called α-condensing if for any bounded set A in X, $f(A)$ is bounded and $\alpha(f(A)) < \alpha(A)$ whenever $\alpha(A) > 0$.

First we prove a common fixed point theorem for pairs of selfmaps of a D-metric space under a certain compactness condition and then derive some interesting fixed point theorems for α-condensing selfmaps of a D-metric space satisfying a contractive condition more general than (3.11).

Theorem 3.2 : Let f and g be two self-maps of a compact D-metric space X satisfying

$$\rho(fx, fy, fz) < max\ \{\rho(gx, gy, z), \rho(gx, fx, z), \rho(gy, fy, z), \rho(gx, fy, z), \rho(gy, fx, z)\}) \qquad (3.15)$$

for all $x, y, \in X$ and $z \in \overline{O(f/g, x) \cup O(f/g, y)}$, for which max $\rho(gx, gy, z) \cdot \rho(gx, fx, z), \rho(gy, fy, z), \rho(gx, fy, z), \rho(gy, fx, z)\} \neq 0$.

Suppose further that

(i) $f(X) \subseteq g(X)$, and $g(X)$ is compact,

(ii) f and g are coincidentally commuting and

(iii) f and g are continuous on g (X).

Then f and g have a unique common fixed point.

<u>**Proof :**</u> If max $\rho(gx, gy, z)\ \rho(gx, fx, z), \rho(gy, fy, z), \rho(gx, fy, z), \rho(gy, fx, z)\}=0$, for some $x, y, z \in X$, then $fx = gx = z$ and by the coincidentally commutativity of f and g, we get $fz = gz$. Consequently z is a common fixed point of f and g. To see this if $z \neq gz$, then

$\rho(z, gz, z)$

$= \rho(fx,, fz, z)$

$< max\ \{\rho(gx, gz, z)\ \rho(gx, fx, z), \rho(gz, fz, z), \rho(gx, fz, z), \rho(gz, fx, z)\}$

$= \rho(z, gz, z)$

which is a contradiction and so $z = gz = fz$.

The uniqueness follows from condition (3.15).

Now suppose that max $\{\rho(gx, gy, z)\ \rho(gx, fx, z), \rho(gy, fy, z), \rho(gx, fy, z), \rho(gy, fx, z)\} \neq 0$ for all $x, y \in X$ and $z \in g(X)$. Obviously $f, g: g(X) \to g(X)$

Define the function $T: g(X) \times g(X) \times g(X) \to (0, \infty)$ by

$$T(x,y,z) = \frac{\rho(fx,fy,z)}{max\{(gx,gy,z),\rho(gx,fx,z),\rho(gy,fy,z),\rho(gx,fy,z),\rho(gy,fx,z)\}} \quad (3.16)$$

for $x, y \in X, z \in O(f/g,x) \cup O(f/g,y)$

Clearly T is well defined since the denominator in (3.16) is not zero for all $x, y \in X, z \in O(f/g,x) \cup O(f/g,y)$. As f and g are continuous, T is continuous and from the compactness of $g(X)$ and $\overline{z \in O(f/g,x) \cup O(f/g,y)}$ it follows that T is a bounded and attains its maximum on $[g(X)]^3$. Call the value c. from (3.15), we get $0 < c < 1$. Hence by definiton of T,

$$\begin{aligned} &\frac{\rho(fx,fy,z)}{max\{\rho(gx,gy,z),\rho(gx,fx,z),\rho(gy,fy,z),\rho(gx,fy,z),\rho(gy,fx,z)\}} \\ &\leq T(u,v,w) \\ &= c \end{aligned}$$

i.e. $\rho(fx,fy,z) \leq c\,max\{\rho(gx,gy,z),\rho(gx,fy,z),\rho(gy,fy,z),\rho(gx,fy,z),\rho(gy,fx,z)\}$

for $x, y \in X, z \in O(f/g,x) \cup O(f/g,y)$.

Since $g(X)$ is compact, it is bounded and complete. Now the desired conclusion follows by an application of theorem 3.1. This completes the proof.

Corollary 3.5 : Let f and g be two self-maps of a D-metric space X, p and m the positive integers, satisfying

$$\rho(f^px,f^py,z) < max\{\rho(g^mx,f^py,z),\rho(g^mx,f^px,z),\rho(g^my,f^py,z),\rho(g^mx,f^py,z),\rho(g^my,f^px,z)\}$$

(3.17)

for all $x,y \in X$ and $z \in O\left(f^m / g^m, x\right) \cup O\left(f^m / g^m, y\right)$

if $max\left\{\rho\left(g^m x, g^m y, z\right), \rho\left(g^m x, f^p x, z\right), \rho\left(g^m y, f^p y, z\right), \rho\left(g^m x, f^p y, z\right), \rho\left(g^m y, f^p x, z\right)\right\} \neq 0$
Further suppose that

(i) $f^p(X) \subseteq g^m(X)$ *and* $g^m(X)$ *and* $g^m(X)$ is compact,

(ii) f and g are commuting and

(iii) f^p and g^m are continous on $g^m(X)$

Then f and g have a unique common fixed point.

Proof: The proof is similar to the Corollary 3.1 and now the conclusion follow by an application of Theorem 3.2. This completes the proof.

Corollary 3.6: $f: X \to X, X$ a f-orbitally complete D-metric space be a f-orbitally continuous and α-condensing map satisfying

$$\rho(fx, fy, z) < max\{\rho(x,y,z), \rho(x,fx,z), \rho(y,fy,z), \rho(x,fy,z), \rho(y,fx,z)\} \qquad (3.18)$$

for all $x,y \in X$ and $z \in \overline{O(x) \cup O(y)}$ for which

$max\{\rho(x,y,z), \rho(x,fx,z), \rho(y,fy,z), \rho(x,fy,z), \rho(y,fx,z)\} \neq 0$. Then f has a unique fixed point, u and f is continous at u.

Proof : Let $x \in X$ be arbitary and define a sequence $\{x_m\} \subset X$ by

$$x_0 = x, x_{n+1} = fx_n, \; n \geq 0.$$

Let $A = \{x_0, x_1, \ldots\ldots, x_m \ldots\ldots\}$. We show that A is precompact. Suppose not.

Then $\alpha(A) > 0$ and from the properties $(\alpha 1) - (\alpha 4)$ it follows that

$$\alpha(A) = \alpha(\{x_0, x_1, \ldots\ldots\})$$

$$=\alpha(\{x_0\}\cup f(A))$$

$$=\alpha(f(A))$$

$$<\alpha(A).$$

This is a contradiction. Hence $\underline{A}$ is precompact and $\overline{A}$ is compact, since A is complete. Obviously $f:\overline{A}\to\overline{A}$. Now the desired conclusion follows by an application of Theorem 3.2 with $X=\overline{A}$ and $g=I$, the identity map on $\overline{A}$.

Corollary 3.7 : Let f be a f-orbitally continous and α-condensing self-map of a f-orbitally complete D-metric space X, satisfying for some positive integer p,

$$\rho(f^px,f^py,z)<max\{\rho(x,y,z),\rho(x,f^px,z),\rho(y,f^py,z),$$
$$\rho(x,f^py,z)\rho(y,f^px,z)\} \tag{3.19}$$

for all $x,y\in X$ and $z\in O(x)\cup O(y)$, if

$max\{\rho(x,y,z),\rho(x,fx,z),\rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\}\neq 0$. Then f has a unique fixed point u and f^p is continuous at u.

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GEOMERTY IN RITUALS

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ABSTRACT

Śulbasūtras, the manuals for constructing sacrificial altars of various geometrical designs, are the rich sources of mathematical knowledge of Indo-Aryan savants of first millennium B.C. at least. This paper highlights the major contents of the ***Śulbasūtras***.

MATHEMATICS SUBJECT CLASSIFICATION : 01A32.

1. INTRODUCTION

It is a well established fact that the cities of the Harappa and Mohenjodaro of Indus valley civilization were highly developed. Architecture and town planning in this civilization seem to be based on high class knowledge. Different types of stone weights and nicely finished metals of different shapes excavated in these cities are the testimony of a widely spread advanced scientific knowledge and human skill ever had. Various designs such as squares, rectangles, intersecting or superimposed circles, circles with 4, 8 or 12 spokes, concentric circles and squares, rhombi, triangles, single or double crosses, criss-cross and other figures including complex ones drawn on ornaments, seals and other objects certainly focus their geometrical knowledge of high order. See [5] and [10] for details.

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GEOMETRY IN VEDIC TRADITIONS

The Saṃhitas : Each of the four *Vedas, Ṛg-veda* (divine source of all knoweldge), *Yajur-veda* (dealing with religious rites), *Sama-veda* (comprising of music and fine arts) and *Atharva-veda* (science of medicine), is associated with different schools known as *Saṃhitas* or recensions of *Veda*. The number of such schools according to Patan˜jali (ca. 150 B.C.), the commentator of Paṇini's grammar (ca. 500 B.C.), is described in the following stich :

एकविंशतिधा वा ह्वृच्यम्, एकशतमध्वर्य्यशाखाः।
सहस्रवर्त्मा सामवेदः, नवधा आथर्व्वणो वेदः पञ्चदशभेदो वा॥

That is: There were 21 different schools of the *Ṛg-veda*; 101 schools of the *Yajur-veda;* 1000 of the *Sāma-veda* and 9 or 15 of the *Atharva-veda* ([3, p.1]).

The Associated Literature: These are: *Brāhmaṇas* (theological and ritual treatises), *Āṛṇyakas* (metaphysical appendices of *Brāhmaṇas*) and *Upaniṣadas* (philosophical texts regarding the absolute one).

The Vedāṅ gas: The six *Vedāṅgas* (limbs of *Vedas* are namely, *Śikṣā* (phonetics), *kalpa* (rituals), *Vyākarṇa* (grammar), *Nirukta* (etymology), *Chandas* (prosody) and *Jyotiṣa* (astral science).

Performing *Yajñas* on various religious and ceremonial occasions were part and parcel of *Vedic* savants. Astronomy was developed to carry proper time to carry out a particular sacrifice, and geometry and perhaps also arithmetic and algebra for designing and constructing a large number of *vedis* (sacrifical grounds), *citis* (mounds or altars), *agnis* (fire-places) and *kuṇḍas* (pits) for *yajñas* on this occasion. The procedure for performing sacrifices and ceremonies are spread over *Gṛhyasūtras* of *Kalpa* literature. (In broad sense the Kalpa literature is divided into two portions, viz., *Gṛhyasūtras* and *Śrautasūtras*). The rules relating to construction and transformation of various figures pertaining to shapes of altars and other mathematical details as found in *Śrautasūtras* were later on elaborated in much extent. This sytematized and extended version assumed the form known as *Śulbasūtras* (literal meaning : rules of chord). Thus *Śulbasūtras* are the oldest available geometrical treatises. A. Michels (cf. [5]) has recently displayed that "Vedic geometry, though non-axiomatic in character, is provable and indeed proof is implicit in several constructions perscribed in the *Śulbasūtras*". The nine *Śulbasūtras* so far known are:

Śulbasūtra	**Corresponding *Saṃhitā***
Baudhāyana (the oldest); *Āpastamba; Vādhūla; Hiraṇyakeśin*	*Taittrīya Saṃhitā* of *Kṛṣṇa yajur-veda*
Mānava; Vārāha	*Maitrāyaṇa Saṃhitā* of *Yajur-veda*
Laugākṣi	*Kāṭhak-kapiṣṭhala Saṃhitā* of *Yajur-veda*
Kātyāyana	*Śukla Yajur-veda*
Maśaka	*Sāma-veda*

Among these, the important and existing *śulbasūtras constitute* Āpastamba, *Baudhāyana, Kātyāyana and Mānava* Śulbasūtras.

Baudhāyana Śulbasūtra (BSS)

It is the largest of the *śulba* compositions consising of three chapters (*adhyāyas*). Chapters 1-3 contain 116, 86 and 323 *sūtras* (aphorisms) respectively.

Commentators:

Dvārkānātha Yajvā ("Baudhayāna Śulbasūtravyākhyā " or *"Śulbadīpikā")*

Vyaṅkaṭeśvara Dīkṣita (*"Śulbamīmāṃsā")*

Anonymous commentary: *Baudhāyana śulbasūtra-vyākhyā*

Refer [3] and [4] for further details.

***Āpastamba Śulbasūtras* (*ASS*)**

It is classified into six sections (*paṭalas*) and each section into a number of chapters. The number of chapters inolved from the first to the last section are in the order 3, 4, 3, 4, 3 and 4 respectively. It includes a total of 223 *sūtras*.

Commentators:

Gopāla Yajvā ("Āpastamba Savyākhyā")
Kapardisvāmī ("Āpastamba-śulbaśūtrabhāṣya")
Karvindasvāmi ("Āpastamba Savyākhyā")
Sundararāja ("Āpastamba Savyākhyā")

Sundara Sūri ("*Āpastambaśulbaśūtrabhāṣya*")

For details, refer to [3, p. 3-4], [9, p. 23-34], [13, p. 20-30] and [14, p. 5].

Kātyāyana Śulbasūtras (KSS)

It is divided into two parts. The first part composed in the form of *sūtras* consists of seven small sections (*kaṇḍikās*) and contains 90 *sūtras*, while the second part composed in verses contains nearly 40 verses.

Commentators:

Gaṅgādhara ("Kātyāyana-śulbabhāṣysa")
Karka ("*Kātyāyana-śulbasūtravivaraṇa")*
Mahīdhara ("*Kātyāyana-śulbavivṛti")*
Ramācandra ("*Kātyāyana-śūlbavyākhyā*")

For details, see [3, p. 4-5] and [14, p. 107].

Manava Śulbasūtras (MSS)

This is a small treatise composed in both prose and verse, and comprises of seven sections *(khaṇḍas).*

Commentator:

Śivadāsa ("Mānava-śulbhāṣya")
Anonymous commentary: *Mānavaśulbaśūtrabhāṣya*
Refer [14, p. 142].

Here is a brief description of demonstration while performing *eccelestial* rites as discussed in the *Śulbasūtra* (cf. [1, pp. 5-6]):

The ceremonies were performed on the top of the altars either in sacificer's house or on a nearby plot of ground. The altar is a specified raised area, generally made of bricks for keeping fire. The fire altars were of two types, nemely: the perpetual (daily) and optional (wish fulfilment). The perpetual fires were constructed on a smaller area of one square *puruṣa* and the optional fires were constructed on a bigger

area of $7\frac{1}{2}$ square *puruṣa* or more, each having minimum of live layers of bricks. The perpetual fires had 21 bricks and optional fires had 200 bricks in each layer in the first construction, and the number of bricks become more in the subsequent constructions with various other restrictions. The measurement of area and the construction of bricks for optional fires were a laborious process and the ceremonies continued for a long time. The whole family of the organiser had to reside by the side of the optional fire altar, for which another class of structure known as *Mahāvedi* and other related *vedis* were made. The tradiation is very old in India and detailed information are available in the *saṃhitās*, *Brāhmaṇas* and *śulbasūtras.*

CHRONOLOGY OF VEDIC LITERATURE.

A section of prominent Vedic scholars hold Veda(s) as beginingless (indeed as old as this creation) (see, for instance, *Svāmī Dayānanda Sarasvatī's Ṛg-vedādi Bhāṣya Bhumikā*, Ajmer, and [18]). Following Keith and Satyaprak*ash* [12, p. 5-6] and [4, p. 1-2]. Vedas, the earliest axtant compositions, and subsequent Vedic literature are assigned the following dates:

Vedas	300*0* B.C.
Brāhmaṇas	*2500-2000* B.C.
BSS	800 B.C.
ASS	600 B.C.
MSS	550 B.C.
Paṇini Sūtra	500 B.C.
KSS	200 B.C.

Two Recent Investigations

The recent deciphering of Harppan Language by Louisiana State University Professor and Cryptologist, S. Kak suggests the period of Vedas around 8000 B.C. [18]. A deep analysis by Siddharth [16] under the litle "*Mahāyuga:* The great cosmic

cycle and the date of *Ṛg-veda*" concludes *Ṛg-veda* to be at least as old as 7300 B.C. If we accpet 8000 B.C. for *Ṛg-veda,* the other dates of *Brāhmaṇas* and *Śulbasūtras* should also be pushed back proportionately. In a more recent attempt to dig astronomical proof from *Ṛg-veda*, Siddharth [17] places this date back to atleast 10000 B.C.

HIGHLIGHTS OF ŚULBASŪTRAS

Contents	**Sūtra reference**
-Units of linear measurement	:*ASS* 6.5, 15.4
	BSS 1.3
	KSS 2.1
	MSS 2.1, 11.1 - 11.8, 44, 46
- Surd (irrational numbers)	:*ASS* 2.2-2.3
	BSS 1.9 - 1.11
$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} + \frac{1}{3.4.34}$ (appr.)	*ASS* 1.6
	BSS 2.12
	KSS 2.9
- Theorem of square on the diagonal	:*ASS* 1.4
	BSS 1.2
	KSS 2.7
	MSS 10.10
- Values of π	:*BSS* 4.15
	MSS 11.13

Constructon of geometrical figures:

- Square	: *ASS* 1.7, 2.1
	BSS 1.4, 1.5, 1.8
	KSS 1.3
	MSS 1.11 - 1.12
- Rectangle	: *BSS* 1.6
	KSS 1.3
- Isosceles trapezium	: *BSS* 1.7
	KSS 2.10 - 2.12
- Drawing of east-west and north- south lines	: *KSS* 1.2
Drawing of east-west line	: *MSS* 1.2-1.3
- Rhombus	: *BSS* 2.8
	KSS 4.4
- Triangle	: *BSS* 2.7
	KSS 4.3

Transformation of geometrical figures:

- Square equal to *n* times the given square	: *KSS* 6.7
- Square equal to the sum of two different squares	: *ASS* 2.4
	BSS 2.1
	KSS 2.13

- Square equal to the difference of two different squares	: *ASS* 2.5
	BSS 2.2
	KSS 3.1
- Square into rectangle	: *ASS* 3.1
	BSS 2.3 - 2.4
	KSS 3.4
- Square/rectangle into isosceles trapezium	: *BSS* 2.6
- Square into circle	: *ASS* 3.2
	BSS 2.9
	KSS 3.11
	MSS 1.8, 11.9 - 11.10
- Rectangle into square;	: *ASS* 2.7
	BSS 2.5
	KSS 3.2 - 3.3
- Rectangle/square into triangle/rhombus	: *BSS* 2.7-2.8
- Circle into square	: *ASS* 3.3
	BSS 2.10 - 2.11
	KSS 3.12
- Rhombus into square	: *KSS* 4.6
- Triangle into square	: *KSS* 4.7
- Isoceles triangle into square	: *KSS* 4.5

Name of optional fire altars:	Horizontal section/shape	*Sūtra* reference	
- *Caturaśra-śyenacit*	Hawk bird with square body, wings and tail rectangles	: *ASS*	10.1 - 10.6
			11.1 - 11.11
		BSS	8.1 - 8.18, 9.2 - 9.10
		MSS	6.11 - 6.15
- *Vakrapakṣavyasta puccahaśyenacit*	Hawk bird with bent wings and out spread tail	: *ASS*	15.2 - 17.10
			18.1 - 20.12
		BSS	10.1 - 10.14
			11.2 - 11.3
- *Kaṅkacit*	Hawk bird with curved wings and tail	: *BSS*	12.1 - 12.8
- *Alajacit*	Alaja bird with curved wings and tail	: *BSS*	13.1 - 13.5
- Praugacit	Isosceles triangle	: *BSS*	14.4 - 14.8
- *Ubhayatapraugacit*	Rhombus	: *ASS*	12.7 - 12.8
		BSS	15.2 - 15.6
- *Rathacakracit*	Chariot wheel	: *ASS*	12.9 - 12.10
		BSS	16.3 - 16.5
			16.6 - 16.20
- *Droṇacit*	Square trough	: *ASS*	13.4 - 13.16
		BSS	17.1 - 17.12
		KSS	4.1 - 4.2
	Circular trough	: *BSS*	18.1 - 18.15
- *Śamaśānacit*	Isosceles trapezium	: *BSS*	19.1 - 19.11

- *Kūrmacit*	Tortoise	: *BSS* 20.1 - 20.21 21.2 - 21.13

(Each optional fire has an area of 7.5 square *puruṣa.*)

Perpetual fire altars:

- *Āhavaniya*	Square	: ....
- *Gārhaptya*	Square	: *BSS* 7.4 - 7.7
	Circle	: *BSS* 7.8
- *Dakṣiṇāgni*	Semicircle	: ....

Vedī:

- *Mahāvedī*	Isosceles trapezium	: *BSS* 4.3
Saumikyāvedī	(face a = 24, base b = 30, height c = 36)	: *BSS* 5.1 - 5.2
- *Sautrāmaṇivedi*	Isosceles trapezium $(a = 8\sqrt{3},\ b = 10\sqrt{3}$ $c = 12\sqrt{3}$	: *ASS* 5.8 - 5.9 : *BSS* 3.12 *KSS* 2.10 - 2.12
- *Paitṛkivedi*	Isosceles trapezium (a = 8, b = 10, C = 12) Square having four corners in four coordinate directions	: *BSS* 3.11 *KSS* 2.3 : ....
- *Prāgvaṃśa*	Rectangle (16×12 or 12×10 *prakramas*)	: .

(Units are in *padas* where otherwise state).

The *śulbasūtras* also deal with the following either implicitly or explicitly in connection with the constructions of altars:

1. Construction of a trapezium similar to a given trapezium.
2. Construction of a trapezium similar to a double of its area.
3. Construction of a trapezium similar to third part of its area.
4. Construction of an isosceles trapezium similar to a given trapezium by increasing/diminshing its sides by a certain proportion.
5. Construction of an isosceles trapezium of given area.
6. Drawing of a straight line at right angles to given straight line from a given point on it.
7. A given finite straight line can be divided into any number of equal parts.
8. Construction of a square of given area.
9. Drawing of a square equivalent to n^{th} part of a given square.
10. Drawing of a square equivalent to two given triangles.
11. Drawing of a square equivalent to two given pentagons.
12. The maximum square that can be inscribed within a circle is the one which has its corners on the circumference of the circle.
13. A circle can be divided into any number of parts by drawing diameters.
14. Construction of a *vedi* similar to a given falcon shaped *vedi.*
15. Construction of a parallel sides at a give inclination.
16. A parallelogram and a rectangle which are on the same base and within the same parallels are equal to one another in area.

17. Each diagonal of a rectangle bisects it.
18. The diagonals of a rectangle bisect one another and they divide the rectangle into four parts; two and vertically opposite are identical in all respects.
19. The diagonals of a rhombus bisect each other at right-angles.
20. An isosceles triangle is divided into two equal halves by the line joing the vertex with the middle point of the opposite side.
21. An isosceles triangle formed by joining the extremities of any side of a square to the middle point of the opposite side is equal to half the square.
22. A triangle can be divided into a number of equal and similar parts by dividing the sides into an equal number of parts and then joining the parts of division two and two.
23. A quadrilateral formed by the lines joining the middle parts of the sides of a square is a squae whose area is half that of the original one.
24. A quadrilateral formed by the lines joining the middle points of the sides of the rectangle is a rhombus whose area is half that of the rectangle.
25. Areas of simple figures such as triangle, rectangle, square, isosceles trapezium, circle and parallelogram.
26. Volume of prism or cylinder and approximate volume of frustrum of pyramid.

Besides many geometrical problems, *Śulbasūtras* are concerned extesively with the use of instruments such as bamboo rod, geometrical compass, peg, rope, *Śaṇku* and *sphya* (wooden sword) for altar construction; relative distance between *Āhavanīya, Dakṣiṇāgni, Gārhapatya and Uttaravedi;* relative position of various altars with regard to *Mahāvedi;* enlargement of fire altars; a few indeterminate problems; a little of geometrical algebra; fractions; elementary concept of arithmetic series. It is remarkable that various rules imply a set of approximate values of irrational numbers of the type $\sqrt{N}$ and π.

Table 1: Surd Values

Serial no.	(Implied) values	References/Remarks
1.	$\sqrt{2} = 1 - \frac{1}{3} - \frac{1}{3.4} - \frac{1}{3.4.34} = \frac{577}{408}$ $= 4.41421$	ASS 1.6, BSS 2.12 & KSS 2.9
2.	$\sqrt{2} = 10/7 = 1.4287.....$ $\sqrt{2} = 7/5 = 1.4$	KSS 11.14 [6]
3.	$\sqrt{3} = 26/15 = 1.7333.....$	ASS 3.3, BSS 2.11 & KSS 3.12 (For rationale of these urles, see Datta [3, p. 147].)

Table 2: Values of π

Serial no.	Rule	(Implied) values	References/Remarks

Circling the square

1.	$R = \frac{L}{6}(2+\sqrt{2})\left(\frac{6}{2+\sqrt{2}}\right)^2$	$= 3.08830...$ using $\sqrt{2} = \frac{577}{408}$	ASS 3.2, BSS 2.9,
		$9(2-\sqrt{2})^2 = 3.08828...$ using $\sqrt{2} = \frac{577}{408}$	KSS 3.11, MSS 1.8 & 11.10
		$\left(\frac{6}{2+\sqrt{2}}\right)^2 = \frac{900}{289} = 3.11418...$ taking $\sqrt{2} = \frac{7}{5}$	

$= \frac{49}{16}$ 3.0625... taking $\sqrt{2} = {}^{10}\!/_{7}$

$9(2-\sqrt{2})^2 = 3.24$ applying $\sqrt{2} = {}^{7}\!/_{5}$

2.9391 applying $\sqrt{2} = {}^{10}\!/_{7}$

2.	$R = \frac{2\sqrt{2}}{5} L$	${}^{25}\!/_{8} = 3.125$	MSS 11.25

49/16 = 3.0625 using $\sqrt{2} = {}^{10}\!/_{7}$

625/195 = 3.2051... using $\sqrt{2} = {}^{7}\!/_{5}$

<u>Squaring the circle</u>

1.	L = (13/15) D	676/225 = 3.0044...	ASS 3.3, BSS 2.11, KSS 3.12
2.	$L = \frac{\sqrt{3}}{2} D$	3	New interpretation of MSS 11.9-11.10 by Hayashi [8]
3.	$L = D\left(1 - \frac{1}{8} - \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right) = \frac{4985}{11136} D$	$= (9785/5568)^2 = 3.08832$	BSS 2.10
4.	L = (8/9) D	$(16/9)^2 = 3.16049$	MSS 1.27 by Datta

[3, p. 149] withou quoting reference. Commentary on KSS by Mahidhara (cf. [1]).

Other values

1.	$C = 3D$	3	BSS 4.15
2.	$C = (16/5)D$	$16/5 = 3.2$	MSS 11.12; MSS 13.6 due to new interpretation of Gupta [6]
3.		$(144/83)^2 = 3.01...$ $512/169 = 3.03 ...$	New interpretation by Gupta [6] in illustration to example of MSS 1.8 from the commentary by Śivadāsa.

[L stands for side-length of a square, and C for circumference and R (D) for radius (diameter) of the circle.]

Table 3

Sundarāja's and others improvements over circle-square conversions (see [7] for details].

Serial no.	Rule	Implied values	Remarks (if any)
1.	$R = \frac{L}{6}(2+\sqrt{2})\left(1-\frac{1}{118}\right)\left[\frac{6x118}{(2+\sqrt{2})117}\right]^2$	$= 3.141325....$ using $\sqrt{2}=\frac{577}{408}$ $\frac{27848}{1521}\left(3-2\sqrt{2}\right) = 3.141329...$	
2.	$L = \frac{13}{15}\left(1+\frac{3}{113}\right)D$	$\frac{12503296}{3980025} = 3.14151$	....
3.	$L = \frac{8}{9}(1-\frac{1}{332})D$	$\frac{1752976}{558009} = 3.14148$	....
4.	$L = \frac{8}{9}(1-\frac{1}{330})D$	$\frac{3097600}{986049} = 3.14143$	

5. $L = \frac{8}{9}(1 - \frac{1}{311})D \quad \frac{3097600}{986049} = 3.14143 \ ....$ By Gupta [7]

6. $L = \frac{9785}{11136}(1 + \frac{1}{117})D \quad = 3.14151 \ ....$ By Gupta [7]

7. $L = \frac{9785}{11136}(1 + \frac{1}{226})D \quad = 3.14151 \ ....$ By Datta [3]

Note: Conversion rules give approximate values. *Sundararāja* (ca. 1500 A.D.) in his Commentary on *ASS* might have obtained the correction factors by utilizing *Āryabhaṭa* I's value $\pi = 62832/20000$ and then applying the ancient approximation $\sqrt{a^2 + r} = a + r/(2a+1)$ [7].

APPENDIX

Ārchaeological surveys during excavations at Mohenjodaro give evidence of at least 5000 years old and highly cultured Indus civilization of Indian subcontinent. Unfortunately, we don't have Indian mathematical document from that age except acknowledging its highly developed system of weights and measures employed in building and geometrical constructions. Later the country was occupied by Āryan invaders, according to modern researchers, who introduced cast system and developed the Sanskrit literature. However a prominent section of scholars from India and abroad refutes any such invasion and is of view that such story was created and spread by foreign rulers in India through various missionaries established under the patronage of East-India Company since nineteen's of eighteenth century A.D. and by others.

For systematic study of history of Indian Mathematics, it would be rather convenient to study in three phases, namely; ancient (3000 B.C. - 500 A.D.), medieval (500 A.D. - 1600 A.D.) and modern (1600 A.D. and onwards).

"Of the two river systems, that of the Indus, now mainly in Pakistan, had the earliest civilization, gave its name to India. The Fertile Plain of the Panjab ("Five Rivers"), watered by the five great tributaries of the Indus - the Jhelam, Chenab, Ravī, Beas and Satlyj - had a high culture over two thousand years before Christ, which spread down the lower Indus, in the Pakistan province of Sind, now passes through barren desert, though this was once a well watered and fertile land.

The basin of the Indus is divided from that of the Ganges by the Thar, or desert of Rajasthan, and by low hills. The watershed, to the north-west of Delhi, has been the scene of many bitter battles since at least 1000 B.C. The western half of the Ganges plain, from the region around Delhi to Patna, and including the Doab, or the land between the Ganges and its great tributary river Yamuna, has always been the heart of India. Here, in the region once known as *Āryāvarta*, the land of the *Āryān*, her ciassical culture was formed" [cf. 2, p. 1-2.]

The Indian subcontinent was known as Jambūdvīpa (the continent of the Jambu tree) or Bhārtavarṣa (the land of the son of Bharata, a legendary emperor). Bhāratavarṣa alone was claimed to be 9000 yojans across and whole of the region (Jambūdvīpa) 33000 (100000 *Yojans* according to a few sources) [op. cit., pp. 1 & 489].

(Implied) values of $\sqrt{N}$ and π from the Śulbasūtras (including various commentaries and others' interpretations).

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